THE ALGEBRAIC COMPATIBILITY OF RIEMANNIAN OPERATORS' VOICE-LEADING PROPERTIES

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Abstract. In the early formulations of Riemannian music theory, the Riemannian operators were defined at least in part by their voice-leading properties. However, some have suggested a root-intervallic approach to the operators. This has the advantage of crystallizing the operators' algebraic properties, but has the disadvantage of abandoning their voice-leading properties. In this paper, we show that there exist classes of pc-set classes for which it is possible to define automorphisms on the Riemannian group that preseve their voice-leading properties.

In [2], Hook defines Riemannian Operators using root-interval motion. This has the advantage of the operators being easily generalizable to arbitrary pc-set classes. However, the voice-leading properties of the operators, such as common-tone retention, are lost when not working with major and minor triads. For example, the *leittonwechsel* operator, when considered as a uniform triadic transformation acting on the pc-set class 3-11, becomes W_4. But, W_4 does not have the hallmark double-common tone retention voice-leading property when applied in other pc-set classes. (For example, see [3])

However, one could argue that the double-common tone retention of the *leittonwechsel* is almost a defining characteristic of that operator, as originally conceived by Riemann and later others (such as [1] and [4]). Hence, it would seem that while the *leittonwechsel* coincides with W_4 on the pc-set class 3-11, perhaps it should coincide with other operators on other pc-set classes. Indeed, the root-interval motion of the leittonwechsel operator on the pc-set class 3-11 appears to be its least interesting aspect. The most interesting aspect is its voice-leading properties and its preservation of the intervals.

Put another way, the voice-leading properties of the Riemannian operators, when defined root-intervallically, are not inherent in themselves. For example, the common-tone retention properties of the Riemannian operators change depending on which pc-set class the operators act on. But is there some pattern or structure to the way that this property changes when moving from one pc-set class to another?

In this paper, we show that if we move from one pc-set class to another, there exists an automorphism on the Riemannian group that preserves the voice-leading properties of the operators, as long as the two pc-set classes involved are "dual" to each other in a way that will be defined later. Along the way, we will also provide a definitive proof to the notion that the root-intervallic approach and the voice-leading approach to Riemannian operators are equivalent in some way.

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In order to make this concept precise and rigorous, it will be necessary to deal simultaneously with operators that are defined root-intervallicaly (as in [2]) and operators that are defined according to their common-tone retention properties (as in [1]). Hence, for the purposes of this paper, the term *Riemannian Group* shall refer to the subgroup of the uniform triadic transformations as defined by Hook, and we shall use the term *Retention Group* for the group that is generated by double common-tone retention operators.

Indeed, much of the literature in Riemannian theory seem to conflate these two groups. Indeed, the Riemannian group and the retention group is often considered as different approaches to the same object. In fact they are mathematically different objects, although we shall prove in section 3 that they are, in fact, isomorphic. But the situation is more subtle- the isomorphism is not canonical. This means that the voice-leading properties of the Riemannian operators change depending on the underlying pc-set class, as already stated.

In [2], the retention group in the pc-set class of major and minor triads are taken as motivation for defining the Riemannian group. This paper will show how the Riemannian group relates to the retention group in other pc-set classes.

In sections 1 and 2, we provide a rigorous exposition of the already-known fact that different choices in the root or the canonical representative in an arbitrary pc-set class correspond to a "relabeling" of the Riemannian operators. But more importantly, this section introduces key notation used in sections 3 and 4, and also presents a theorem that simplifies the proof of the main result in section 4.

In section 4, we define the notion of dual pc-set classes of trichords, and show that the voice-leading properties of the Riemannian operators are algebraically compatible(that is, an automorphism exists that preserves common-tone retention) if and only if the underlying pc-set classes are dual. Here is the statement of the main result, given now for immediacy.

Theorem 0.1 (Voice-Leading Automorphism on the Riemannian Group)**.** *Let γ and γ' be dual pc-set classes with coordinate systems f and f', respectively. Then $\exists ! \Phi : \mathcal{R} \to \mathcal{R}$, an automorphism, such that if the operator $U \in \mathcal{R}$ has n common-tone retention with respect to f on γ, then $\Phi(U)$ also has n common-tone retention with respect to the coordinate system f' on γ'*

1. Coordinate Systems on pc-Set Classes

Throughout this section, $\Gamma = \{(r, \Delta) \mid r \in \mathbb{Z}_{12},\ \Delta \in \{+, -\}\}$. That is, Γ is the space used to represent major and minor triads as in [2]. $\mathcal{R}$ is the group of Riemannian operators on Γ. Also, γ shall denote any pc-set class. If γ has two transposition classes, we write them as γ^+ and γ^- so that $\gamma^+ \cup \gamma^- = \gamma$. Also, we summarize the following basic definitions to fix notation:

Definition 1.1 (Transposition Operators)**.**

(1) *Let* $T_n : \gamma \to \gamma$, *where* $T_n\Big[\{a_0, a_1, a_2, \cdots, a_j\}\Big] = \{a_0 + n, a_1 + n, a_2 + n, \cdots, a_j + n\}$.
(2) *Let* $T^*_n : \Gamma \to \Gamma$, *where* $T^*_n\Big[(r, \Delta)\Big] = (r + n, \Delta)$.

Definition 1.2 (Inversion Operators)**.**

(1) *Let* $I : \gamma \to \gamma$, *where* $I\Big[\{a_0, a_1, a_2, \cdots, a_j\}\Big] = \{-a_0, -a_1, -a_2, \cdots, -a_j\}$.

(2) *Let* $I^* : \Gamma \to \Gamma$, *where* $I^*\Big[(r, \Delta)\Big] = (r, -\Delta)$.

Clearly, the elements of Γ can be used to represent the elements of γ by defining a bijection between the two sets. But not all identifications between Γ and γ are musically significant. Hence, we shall now define a particular type of bijection called a *coordinate system*. While the following definition may seem very general and elegant, we shall soon see that the coordinate systems are precisely the types of representations that are musically significant. In particular, we may think of them as a choice of tonic in a rough sense.

Definition 1.3 (Coordinate System). *Let* $\Gamma = \{(r, \Delta) \mid r \in \mathbb{Z}_{12}, \Delta \in \{+, -\}\}$. *Let* γ *be a pc-set class with* 24 *elements. A coordinate system on* γ *is a bijection* $f : \Gamma \to \gamma$ *such that* $\forall n$,

$$T_n \circ f = f \circ T_n^*.$$

As a passing note, we mention that the above definition bares a striking resemblance to that of the uniform triadic transformations. More specifically, the UTT group can be shown to be the centralizer of the transposition operators.

Now we shall prove that the above definition has very musical properties.

Theorem 1.4. *Suppose that* f *is a coordinate system. Then*

(1) *For any* $r \in \mathbb{Z}_{12}$, $I \circ f\big[(r, \Delta)\big] = f\big[(s, -\Delta)\big]$ *for some* $s \in \mathbb{Z}_{12}$.
(2) *For any* $r \in \mathbb{Z}_{12}$, $T_n \circ f\big[(r, \Delta)\big] = f\big[(r + n, \Delta)\big]$.

Roughly speaking, the above theorem shows that if γ^+ and γ^- are the two transposition classes of a pc-set class, then a coordinate system assigns all of γ^+ to a tuple of the form (r, Δ) and all of γ^- to a tuple of the form $(r, -\Delta)$. Furthermore, a transposition in Γ is precisely the same as the corresponding transposition in γ.

Proof. (2) follows trivially from the definition of coordinate systems.

To show (1), note that because γ is a pc-set class with 24 elements, $I \circ f\big[(r, \Delta)\big] \neq T_n \circ f\big[(r, \Delta)\big]$ for any n. But $T_n \circ f\big[(r, \Delta)\big] = f \circ T_n^*\big[(r, \Delta)\big]$, so $I \circ f\big[(r, \Delta)\big] \neq f \circ T_n^*\big[(r, \Delta)\big]$. In other words, $I \circ f\big[(r, \Delta)\big] \neq f\big[(n, \Delta)\big]$ $\forall n$. Hence, $\exists s$ s.t. $I \circ f\big[(r, \Delta)\big] = f\big[(s, -\Delta)\big]$.

□

We note that choosing a coordinate system for a pc-set class is a notion that generalizes picking a "root" for chords in a pc-set class. For example, representing the major and minor triads as an ordered tuple (r, Δ) where r is the root of the triad and Δ is $+$ or $-$ depending on whether the triad is major or minor, respectively, is a particular coordinate system on the pc-set class 3-11. Or, suppose we represent the major and minor triads as above with the exception that for minor triads, we pick what is traditionally the "fifth" as the root, so that, for example, $(0, -)$ is f-minor. This is the traditional Riemannian scheme, and will be of some importance in section 4.

Note that choosing a coordinate system also may correspond to picking the "tonic". So for example, the coordinate system given above for major/minor triads may correspond to the key of C-Major, but the coordinate system that assign the g-minor triad to $(0, +)$ and the G-Major triad to $(0, -)$ may correspond to the key of g-minor.

2. Preservation Automorphisms

Now, depending on the choice of a coordinate system, the action of a Riemannian operator on the underlying pc-set class space changes, as the Riemannian operators are defined root-intervallicaly. The following important theorem shows that the action of Riemannian operators can be preserved under a change of coordinate system via an automorphism:

Theorem 2.1 (Preservation Automorphism)**.** *Let f and g be coordinate systems on a pc-set class γ. Then there exists an automorphism Ω on $\mathcal{R}$ such that for all $U \in \mathcal{R}$,*

$$f \circ U \circ f^{-1} = g \circ \Omega(U) \circ g^{-1}.$$

In other words, the following diagram commutes:

$$\begin{array}{ccccccc} \gamma & \xrightarrow{f^{-1}} & \Gamma & \xrightarrow{U} & \Gamma & \xrightarrow{f} & \gamma \\ & \overset{g^{-1}}{\searrow} & \Gamma & \xrightarrow{\Omega(U)} & \Gamma & \overset{g}{\nearrow} & \end{array}$$

We shall call Ω the preservation automorphism corresponding to the ordered pair of coordinate transformations (f, g).

We shall delay the proof of this theorem until the preservation automorphisms are given explicitly in a later theorem. In order to give the automorphism explicitly, we need to define how to precisely measure the "difference" between two coordinate systems.

Definition 2.2 (Modulation Function)**.** *Let $\mathcal{C}$ be the set of all coordinate systems on γ. Let $f, g \in \mathcal{C}$. The modulation function $D : \mathcal{C} \times \mathcal{C} \to \Gamma$ is given by $D(f, g) = (r, \sigma)$ where*

1. (a) $\sigma = +$ *if* $I \circ g\big[(x, \Delta)\big] = f\big[(s, -\Delta)\big]$ *for some* s
   (b) $\sigma = -$ *if* $I \circ g\big[(x, \Delta)\big] = f\big[(s, \Delta)\big]$ *for some* s
2. $r = j - k$ *where* $\forall x \in \mathbb{Z}_{12}$,
   $g\big[(x, \sigma)\big] = T_j \circ f\big[(x, +)\big]$ *and*
   $g\big[(x, -\sigma)\big] = T_k \circ f\big[(x, -)\big]$.

Now we define some specific automorphisms on $\mathcal{R}$ that will be useful later. Recall that the Riemannian group $\mathcal{R}$ is generated by the elements S_1 and W_0, where $S_n = (S_1)^n$ and $W_n = (S_1)^n W_0$. Hence, an endomorphism on $\mathcal{R}$ may be fully characterized by its action on these generators.

Theorem 2.3 (Automophisms on $\mathcal{R}$)**.** *Let $r \in \mathbb{Z}_{12}$. Let $\phi^{S_{r/2}}$ be the endomorphism where $S_1 \mapsto S_1$ and $W_0 \mapsto W_r$. Also let $\phi^{W_{r/2}}$ be the endomorphism where $S_1 \mapsto S_{11}$ and $W_0 \mapsto W_r$. Then $\phi^{S_{r/2}}$ and $\phi^{W_{r/2}}$ are automorphisms on $\mathcal{R}$. Furthermore, they are inner automorphisms if and only if r is even.*

Theorem 2.4 (Preservation Automorphisms)**.** *Let f and g be coordinate transformations on a pc-set class γ. Then then preservation automorphism Ω corresponding to (f, g) is given by the following:*

1. *If* $D(f, g) = (r, +)$, *then* $\Omega = \phi^{S_{r/2}}$.
2. *If* $D(f, g) = (r, -)$, *then* $\Omega = \phi^{W_{r/2}}$.

Proof. Fix some $U \in \mathcal{R}$. Then $U = W_n$ or S_n. Furthermore, suppose that $D(f,g) = (r,+)$. We shall show that $f \circ U \circ f^{-1} = g \circ \Omega(U) \circ g^{-1}$ by direct computation. So first suppose that $U = S_n$. We proceed by computing on a typical element of γ:

$$\begin{aligned}
g \circ \Omega(S_n) \circ g^{-1}(\{a,b,c\}) &= g \circ \phi^{S_{r/2}}(S_1^n) \circ g^{-1}(\{a,b,c\}) \\
&= g \circ \phi^{S_{r/2}}(S_1)^n \circ g^{-1}(\{a,b,c\}) \\
&= g \circ S_1^n \circ g^{-1}(\{a,b,c\}) \\
&= g \circ S_n \circ g^{-1}(\{a,b,c\}) \\
&= \begin{cases} T_j \circ f \circ S_n \circ g^{-1}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = + \\ T_k \circ f \circ S_n \circ g^{-1}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = - \end{cases} \\
&= \begin{cases} T_j \circ f \circ S_n \circ f^{-1} \circ T_{-j}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = + \\ T_k \circ f \circ S_n \circ g^{-1} \circ T_{-k}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = - \end{cases} \\
&= \begin{cases} f \circ T_j^* \circ S_n \circ T_{-j}^* \circ f^{-1}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = + \\ f \circ T_k^* \circ S_n \circ T_{-k}^* \circ g^{-1}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = - \end{cases} \\
&= \begin{cases} f \circ S_{j+n-j} \circ f^{-1}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = + \\ f \circ S_{-k+n+k} \circ g^{-1}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = - \end{cases} \\
&= f \circ S_n \circ f^{-1}(\{a,b,c\})
\end{aligned}$$

The computations for $U = W_n$ and $D(f,g) = (r,-)$ are very analogous. □

3. Retention Operators

For the rest of this paper, γ shall always denote pc-set classes of trichords.

Note that once a specific coordinate system for a pc-set class has been chosen, a Riemannian operator induces a function on the pc-set class directly:

$$\begin{array}{ccc} \Gamma & \xrightarrow{U} & \Gamma \\ f\downarrow & & f\downarrow \\ \gamma & \cdots> & \gamma \end{array}$$

We can define the group of operators acting on γ with function composition as the group operation. We shall denote this group by $\mathcal{C}$, relative to the pc-set class γ.

We shall see shortly that the group $\mathcal{C}$ "downstairs" are precisely those generated by the double-common tone retention operators. Hence, we clarify some terminology first. If an operator $U : \gamma \to \gamma$ has the property that for all $a \in \gamma$, $|a \cap G(a)| = n$ and $n \neq 0$, then we shall call G a retention operator and the elements of $a \cap G(a)$ the *retention notes* of G. Furthermore, n shall be called the retention value of G, denoted by $\#_r(G)$. We clarify the above discussion with the following definition:

Definition 3.1 (Retention Value)**.** *Let γ be a pc-set class with a coordinate system f.*

1. *Suppose that $G \in \mathcal{C}$ has the property that for all $a \in \gamma$, $|a \cap G(a)| = n$. Then the retention value of G is $\#_r(G) = n$.*
2. *The retention value of $U \in \mathcal{R}$ relative to f is $\#_r^* = \#_r(f \circ U \circ f^{-1})$.*

Hook comments that the voice-leading properties of the Riemannian operators (that is, the retention values of the operators) change when moving from one pc-set class to another. However, we shall soon see that we can define the notion of dual pc-set classes, and that the retention properties of the Riemannian operators can be preserved via an automorphism if and only if one moves from a pc-set class to its dual.

To illustrate this, consider an operator G on the pc-set class γ induced by a Riemannian operator U. Now apply the same operator G wit the same voice-leading properties on γ'. This, in turn, induces a Riemannian operator U' as the following diagram demonstrates:

$$\begin{array}{ccccccc} \Gamma & \xrightarrow{U} & \Gamma & & \Gamma & \overset{U'}{\cdots>} & \Gamma \\ f\downarrow & & f\downarrow & \Rightarrow & g\downarrow & & g\downarrow \\ \gamma & \overset{\cdots>}{G} & \gamma & & \gamma' & \overrightarrow{G} & \gamma' \end{array}$$

It turns out that this mapping from U to U' is an automorphism on $\mathcal{R}$ if and only if the underlying pc-set classes are dual to each other. Note that in the special case where $\gamma = \gamma'$, the above situation is precisely a change of coordinate systems.

But first, we shall work to show that $\mathcal{C}$ is precisely the group generated by the operators with common-tone retention.

Definition 3.2 (Retention Operators)**.** *Fix some coordinate system f on γ. Also fix some chord $\{a, b, c\} \in \gamma$. (Sometimes, it is convenient to think of {a,b,c} as the prime form of the pc-set class, but this is not a logical necessity.)*

1. *The retention operator $R_{a,b} : \gamma \to \gamma$ shall be the operator $f \circ U \circ f^{-1}$ where U is the unique Riemannian operator $U : \Gamma \to \Gamma$ such that $f \circ U \circ f^{-1}(\{a, b, c\}) = \{a, b, x\}$ where $x \neq c$.*
2. *The retention operator $R_a : \gamma \to \gamma$ shall be the operator $f \circ W_n \circ f^{-1}$ where W_n is the unique Weschel operator $W_n : \Gamma \to \Gamma$ such that $f \circ W_n \circ f^{-1}(\{a, b, c\}) = \{a, x, y\}$ where $x, y \neq b, c$.*

For convenience, we shall sometimes write $R_{a,b} \sim U$ to mean $R_{a,b} = f \circ U \circ f^{-1}$ and $R_a \sim W_n$ to mean $R_a = f \circ W_n \circ f^{-1}$.

This definition perhaps best captures the interplay between the voice-leading properties and its Riemannian dualism of retention operators.

Also note that the above definition slightly abuses notation, as the operator $R_{a,b}$ has as its retention notes $\{a, b\}$ if and only if it acts on the chord {a,b,c}. However, $R_{a,b}$ always has two retention notes as the Riemannian operators are uniform (See [2]) .

We begin with a technical lemma that will be the workhorse of the rest of this paper.

Lemma 3.3. *Suppose that γ is a pc-set class with a coordinate system f. Let $\{a, b, c\} \in \gamma$ and $f^{-1}(\{a, b, c\}) = (r, +)$ for some r.*

1. *Let $R_{b,c} \sim W_i$, $R_{a,c} \sim W_j$, and $R_{a,b} \sim W_k$. Then*

$$i - j = b - a$$
$$j - k = c - b$$
$$k - i = a - c$$

(2) *Also let* $R_a \sim W_l$, $R_b \sim W_m$, *and* $R_c \sim W_n$. *Then*

$$l - m = 2(a - b)$$
$$m - n = 2(b - c)$$
$$n - l = 2(c - a)$$

If $f^{-1}(\{a, b, c\}) = (r, -)$, *then simply add a factor of* -1 *to one side of each equation given above.*

As an interesting consequence of the above lemma, it is always true that $\{i, j, k\} \in \gamma$.

Proof. Part 1. We shall show that $i - j = b - a$ if $f^{-1}(\{a, b, c\}) = (r, +)$ and $i - j = -(b - a)$ if $f^{-1}(\{a, b, c\}) = (r, -)$. Then by symmetry, the other equations in (1) also follow by precisely the analogous arguments.

Note that $R_{b,c}(\{a, b, c\}) = \{b, c, c + b - a\}$ and $R_{a,c}(\{a, b, c\}) = \{a, c, c - b + a\}$ simply by the interval structure of γ. Hence, $f \circ W_i \circ f^{-1}(\{a, b, c\}) = \{b, c, c + b - a\}$ and $f \circ W_j \circ f^{-1}(\{a, b, c\}) = \{a, c, c - b + a\}$. Now, consider the case where $f^{-1}(\{a, b, c\}) = (r, +)$ for some r. Since $T^*_{i-j} \circ W_j[(r, +)] = W_i[(r, +)]$,

$$\begin{aligned} f \circ W_i \circ f^{-1}(\{a, b, c\}) &= f \circ T^*_{i-j} \circ W_j \circ f^{-1}(\{a, b, c\}) \\ &= T_{i-j} \circ f \circ W_j \circ f^{-1}(\{a, b, c\}). \end{aligned}$$

Hence,

$$\begin{aligned} \{b, c, c + b - a\} &= T_{i-j}(\{a, c, c - b + a\}) \\ &= (\{a + i - j, c + i - j, c - b + a + i - j\}. \end{aligned}$$

Now, considering every bijection possible between $\{b, c, c + b - a\}$ and $\{a + i - j, c + i - j, c - b + a + i - j\}$, the only system of equations that does not lead to a contradiction is the one that yields $i - j = b - a$. A similar computation for the case where $f^{-1}(\{a, b, c\}) = (r, -)$ yields $i - j = -(b - a)$.

Part 2. We shall show that $l - m = 2(a - b)$. Then by symmetry, the other equations in (2) also follow by precisely the analogous arguments.

Note that $R_a(\{a, b, c\}) = \{2a - c, 2a - b, a\}$ and $R_b(\{a, b, c\}) = \{b, 2b - a, 2b - c\}$ simply by the interval structure of γ. Hence, $f \circ W_l \circ f^{-1}(\{a, b, c\}) = \{2a - c, 2a - b, a\}$ and $f \circ W_m \circ f^{-1}(\{a, b, c\}) = \{b, 2b - a, 2b - c\}$. Now, consider the case where $f^{-1}(\{a, b, c\}) = (r, +)$ for some r. Since $T^*_{l-m} \circ W_m[(r, +)] = W_l[(r, +)]$,

$$\begin{aligned} f \circ W_i \circ f^{-1}(\{a, b, c\}) &= f \circ T^*_{l-m} \circ W_m \circ f^{-1}(\{a, b, c\}) \\ &= T_{l-m} \circ f \circ W_m \circ f^{-1}(\{a, b, c\}). \end{aligned}$$

Hence,

$$\begin{aligned} \{2a - c, 2a - b, a\} &= T_{l-m}(\{b, 2b - a, 2b - c\}) \\ &= (\{b + l - m, 2b - a + l - m, 2b - c + l - m\}. \end{aligned}$$

Now, considering every bijection possible between $\{2a - c, 2a - b, a\}$ and $\{b + l - m, 2b - a + l - m, 2b - c + l - m\}$, the only system of equations that does not lead to a contradiction is the one that yields $l - m = 2(a - b)$. A similar computation for the case where $f^{-1}(\{a, b, c\}) = (r, -)$ yields $l - m = -2(a - b)$. □

Finally, we arrive at the theorem that shows that the space of operators "downstairs" is isomorphic to the Riemannian group.

Theorem 3.4 (Group Structure of the Voice-Leading Group). *Consider any* $\{a, b, c\} \in \gamma$*. Suppose that* $\gcd(b-a, 12) = 1$*. Then* $R_{b,c}$ *and* $R_{a,c}$ *generates* D_{12}*, the dihedral group on a 12-gon, and hence is isomorphic to* $\mathcal{R}$*. We shall denote this group by* $\mathcal{C}$*.*

This theorem definitively provides conditions for when the voice-leading approach and root-intervallic approach to transformation theory are equivalent in general pc-set classes up to isomorphism. However, it should be noted that this isomorphism is not canonical and certainly does not preserve voice-leading properties in general. We shall have more to say on that matter in the next section.

We note that the only pc-set class of trichords with 24 chords that do not obey the hypotheses of this theorem is the pc-set class 3-8.

Proof. This proof shall proceed by "lifting" upstairs to Γ and computing there. Fix some coordinate system f on γ. Then because γ has 24 chords, there exists $W_i, W_j \in \mathcal{R}$ such that $R_{b,c} = f^{-1} \circ W_i \circ f$ and $R_{a,c} = f^{-1} \circ W_j \circ f$. Because there is a natural isomorphism between the group generated by $\{W_i, W_j\}$ and $\{f^{-1} \circ W_i \circ f, f^{-1} \circ W_j \circ f\}$, $R_{b,c}$ and $R_{a,c}$ generate D_{12} if and only if W_i and W_j does.

But $i - j = \pm(b - a)$ by 3.3. Hence, because $\gcd(b - a, 12) = 1$ by assumption, it also follows now that $\gcd(i - j, 12) = 1$. This implies that W_i and W_j generate D_{12}. □

Corollary 3.5. *Suppose that* γ *follows the hypotheses of the previous theorem. Then the function* $\psi : \mathcal{R} \to \mathcal{C}$ *where* $\psi(U) = G$ *if* $G \sim U$ *(that is,* $G = f \circ U \circ f^{-1}$*) is a group isomorphism.*

4. The Voice-Leading Automorphism and Dual pc-Set Classes

Now we define what we mean by a dual pc-set class. Later on, we shall see that these dual pc-set classes are precisely the ones with algebraically-compatible voice-leading structures.

Definition 4.1 (Dual pc-set Class). *Let* γ *and* γ' *be pc-set classes of trichords. Further suppose that for any* $\{a, b, c\} \in \gamma$ *and* $\{a', b', c'\} \in \gamma'$, $\gcd(b - a, 12) = 1$ *and* $\gcd(b' - a', 12) = 1$*. Then* γ *and* γ' *are dual if*

$$(b - a)(b - c) = (b' - a')(b' - c')$$

and

$$(b - a)(c - a) = (b' - a')(c' - a').$$

The following table shows which pc-set classes are dual to which pc-set classes, in prime form:

pc-set class	dual pc-set class
{0,1,3}	{0,2,5}
{0,1,4}	{0,3,7}
{0,1,5}	{0,1,5}
{0,1,6}	{0,1,6}
{0,2,6}	None

Table 1. Dual pc-set Classes

It is interesting to note that the prevailing pc-set class in the modern Western world, the pc-set class 3-11, has a dual pc-set class with which its voice-leading properties are algebraically compatible.

Theorem 4.2 (Voice-Leading Isomorphism on the Retention Group)**.** *Let γ and γ' be dual pc-set classes, with corresponding groups $\mathcal{C}$ and $\mathcal{C}'$ generated by the retention operators. Then $\exists! \phi : \mathcal{C} \to \mathcal{C}'$, a nontrivial group isomorphism, such that*

$$\#_r \circ \phi = \#_r.$$

Before proving this main theorem, we begin with a definition and a lemma that will simplify the proof of the above theorem. Recall that if f is a coordinate system, then it has the property $T_n \circ f = f \circ T_n^*$.

Definition 4.3 (Riemannian Coordinate System)**.** *Let f be a coordinate system on a pc-set class γ. f is a Riemannian coordinate system if*

$$I \circ f[(0,+)] = f \circ I^*[(0,+)].$$

In other words, if $f[(0,+)] = \{a,b,c\}$, then $f[(0,-)] = \{-a,-b,-c\}$.

Note that the coordinate system on the pc-set class 3-11 that assigns to $(0,+)$ the C-Major triad and to $(0,-)$ the f-minor triad is a Riemannian coordinate system.

Lemma 4.4. *Let f be a Riemannian coordinate system on γ. Let $\{a,b,c\} \in \gamma$ such that $f^{-1}(\{a,b,c\}) = (0,+)$.*

(1) *Let $R_{b,c} \sim W_i$, $R_{a,c} \sim W_j$, and $R_{a,b} \sim W_k$. Then*

$$i = b + c$$
$$j = c + a$$
$$k = a + b$$

(2) *Also let $R_a \sim W_l$, $R_b \sim W_m$, and $R_c \sim W_n$. Then*

$$l = 2a$$
$$m = 2b$$
$$n = 2c$$

If $f^{-1}(\{a,b,c\}) = (0,-)$, then simply add a factor of -1 to one side of each equation given above.

Proof. This proof is very similar to that of 3.3.

Part 1. We shall show that $k = b + a$ if $f^{-1}(\{a,b,c\}) = (0,+)$ and $k = -(b+a)$ if $f^{-1}(\{a,b,c\}) = (0,-)$. Then by symmetry, the other equations in (1) also follow by precisely the analogous arguments.

Note that $R_{a,b}(\{a,b,c\}) = \{a,b,a-c+b\}$ simply by the interval structure of γ. Also, $R_{a,b}(\{a,b,c\}) = f \circ W_k \circ f^{-1}(\{a,b,c\})$. Now, consider the case where

$f^{-1}(\{a,b,c\}) = (0,+)$. Then,

$$\begin{aligned} R_{a,b}(\{a,b,c\}) &= f \circ W_k \circ f^{-1}(\{a,b,c\}) \\ &= f \circ W_k[(0,+)] \\ &= f[(k,-)] \\ &= f \circ T_k^*[(0,-)] \\ &= T_k \circ f[(0,-)] \\ &= T_k(\{-a,-b,-c\}) \\ &= \{-a+k,-b+k,-c+k\} \end{aligned}$$

because f is a Riemannian coordinate system. It now follows that

$$\{-a+k,-b+k,-c+k\} = \{a,b,a-c+b\}.$$

After considering every possible bijection between the two sets, the only system of equations that does not lead to a contradiction is the one that yields $k = b + a$. A similar computation for the case where $f^{-1}(\{a,b,c\}) = (0,-)$ yields $k = -(b+a)$.

Part 2. We shall show that $n = 2c$ if $f^{-1}(\{a,b,c\}) = (0,+)$ and $n = -2c$ if $f^{-1}(\{a,b,c\}) = (0,-)$. Then by symmetry, the other equations in (2) also follow by precisely the analogous arguments.

Note that $R_c(\{a,b,c\}) = \{c, 2c-b, 2c-a\}$ simply by the interval structure of γ. Also, $R_c(\{a,b,c\}) = f \circ W_n \circ f^{-1}(\{a,b,c\})$. Now, consider the case where $f^{-1}(\{a,b,c\}) = (0,+)$. Then,

$$\begin{aligned} R_c(\{a,b,c\}) &= f \circ W_n \circ f^{-1}(\{a,b,c\}) \\ &= f \circ W_n[(0,+)] \\ &= f[(n,-)] \\ &= f \circ T_n^*[(0,-)] \\ &= T_n \circ f[(0,-)] \\ &= T_n(\{-a,-b,-c\}) \\ &= \{-a+n,-b+n,-c+n\} \end{aligned}$$

because f is a Riemannian coordinate system. It now follows that

$$\{-a+n,-b+n,-c+n\} = \{c, 2c-b, 2c-a\}.$$

After considering every possible bijection between the two sets, the only system of equations that does not lead to a contradiction is the one that yields $n = 2c$. A similar computation for the case where $f^{-1}(\{a,b,c\}) = (0,-)$ yields $n = -2c$. □

Now we can begin the proof of 4.2.

Proof of Theorem 4.2. This proof will proceed in stages, reducing each case to the next.

Step 1. First, we show that there exists a unique non-trivial isomorphism $\phi : \mathcal{C} \to \mathcal{C}'$ such that $\#_r = \#_r \circ \phi$ if and only if there exists a uniqe non-trivial automorphism $\Phi : \mathcal{R} \to \mathcal{R}$ such that $\#_r^* = \#_r^* \circ \Phi$ relative to arbitrary coordinate systems f and f' on γ and γ', respectively.

First, suppose $\exists! \phi : \mathcal{C} \to \mathcal{C}'$, an isomorphism, such that $\#_r = \#_r \circ \phi$. Then let $\Phi = \psi^{-1} \circ \phi \circ \psi$, where ψ is the isomorphism from 3.5. Then Φ is an isomorphism as it is a composition of isomorphisms. Furthermore, $\Phi : \mathcal{R} \to \mathcal{R}$ hence is an automorphism.

Now because $\#_r^* = \#_r \circ \psi$ by definition,

$$\begin{aligned}\#_r^* \circ \Phi &= \#_r^* \circ \psi^{-1} \circ \phi \circ \psi \\ &= \#_r \circ \psi \circ \psi^{-1} \circ \phi \circ \psi \\ &= \#_r \circ \phi \circ \psi \\ &= \#_r \circ \psi \\ &= \#_r^*\end{aligned}$$

For the converse existence, let $\phi = \psi \circ \Phi \circ \psi^{-1}$ where ψ is the isomorphism from 3.5. Then ϕ is an isomorphism as it is a composition of isomorphisms.

Now because $\#_r = \#_r^* \circ \psi^{-1}$,

$$\begin{aligned}\#_r^\circ \phi &= \#_r \circ \psi \circ \Phi \circ \psi^{-1} \\ &= \#_r^* \circ \psi^{-1} \circ \psi \circ \Phi \circ \psi^{-1} \\ &= \#_r^* \circ \Phi \circ \psi^{-1} \\ &= \#_r^* \circ \psi^{-1} \\ &= \#_r\end{aligned}$$

Equivalence of uniqueness follows directly from the equivalence of existence.

Step 2. We now show that there exists unique non-trivial automorphism $\Phi : \mathcal{R} \to \mathcal{R}$ such that $\#_r^* = \#_r^* \circ \Phi$ relative to any arbitrary coordinate systems f and f' on γ and γ', respectively, if and only if there exists unique non-trivial automorphism $\Theta : \mathcal{R} \to \mathcal{R}$ such that $\#_r^* = \#_r^* \circ \Theta$ relative to two fixed Riemannian coordinate systems g and g' on γ and γ', respectively, with the property that $g^{-1}(\{a, b, c\}) = (0, +)$ and $g'^{-1}(\{a', b', c'\}) = (0, +)$.

First, suppose that $\exists! \Phi : \mathcal{R} \to \mathcal{R}$ such that $\#_r^* = \#_r^* \circ \Phi$ relative to any arbitrary coordinate systems f and f' on γ and γ'. Then clearly, this is also the case relative to the two fixed Riemannian coordinate systems g and g' on γ and γ' as g and g' are just a specific instance of an arbitrary coordinate system on γ and γ'.

For the converse, let Θ be the automorphism on $\mathcal{R}$ such that $\#_r^* = \#_r^* \circ \Phi$ relative to Riemannian coordinate systems g and g'. Now let f and f' be arbitrary coordinate systems on γ and γ', respectively. Then let Ω be the preservation automorphism from 2.1 relative to the ordered pair (f, g) and Ω' be the preservation automorphism relative to (g', f'). Let $\Phi = \Omega' \circ \Theta \circ \Omega$. Then Φ has the desired properties as you can check.

Step 3. Now it remains to show that $\exists! \Theta : \mathcal{R} \to \mathcal{R}$ such that $\#_r^* \circ \Theta = \#_r^*$ relative to Riemannian coordinate systems g and g' on γ and γ', respectively.

Now since γ and γ' are dual, let $\{a, b, c\} \in \gamma$ where $\gcd(b - a, 12) = 1$ and $\{a', b', c'\} \in \gamma'$ where $\gcd(b' - a', 12) = 1$.

Let g and g' be Riemannian coordinate systems on γ and γ'. Also suppose that $R_{b,c} \sim W_i$, $R_{a,c} \sim W_j$, $R_{a,b} \sim W_k$, $R_a \sim W_l$, $R_b \sim W_m$, and $R_c \sim W_n$ with respect to g. Likewise, suppose that $R_{b',c'} \sim W_{i'}$, $R_{a',c'} \sim W_{j'}$, $R_{a',b'} \sim W_{k'}$, $R_{a'} \sim W_{l'}$, $R_{b'} \sim W_{m'}$, and $R_{c'} \sim W_{n'}$ with respect to g'.

Now we work to construct Θ with the desired properties. Recall that we may completely specify Θ by its action on generators of $\mathcal{R}$. Because of 3.4, W_i and W_j generate $\mathcal{R}$. But because W_i and W_j are also double common-tone retention operators relative to g, they must map to another double common-tone retention operator relative to g'. So let $\Theta(W_i) = W_{\sigma(i)'}$ and $\Theta(W_j) = W_{\sigma(j)'}$, where σ is a permutation on the indices $\{i, j, k\}$. So for example, if σ is the identity permutation, then $\sigma(i)' = i'$. Note that this forces $\Theta(W_k)$ to equal $W_{\sigma(k)'}$ in order to preserve common-tone retention. Now we seek a permutation σ such that Θ is an isomorphism. Then, we must check that such a isomorphism also preserves operators with single common-tone retention.

Now, $W_k = (W_i W_j)^\alpha W_j$ for some integer α. Hence, $(i-j)\alpha + j = k$, which is equivalent to $\alpha(i-j) = (k-j)$. Now, because $\gcd(i-j, 12) = 1$ and all units in the ring $\mathbb{Z}_{12}$ have the property that $x^{-1} = x$, $\alpha = (k-j)(i-j)$.

Hence, if $\Theta(W_i) = W_{\sigma(i)'}$ and $\Theta(W_j) = W_{\sigma(j)'}$, then

$$\begin{aligned}\Theta(W_k) &= \Theta\big((W_i W_j)^{(k-j)(i-j)} W_j\big)\\ &= \big((W_{\sigma(i)'} W_{\sigma(j)'})^{(k-j)(i-j)} W_{\sigma(j)'}\big)\end{aligned}$$

But, we also have $\Theta(W_k) = W_{\sigma(k)'}$ in order to preserve double common-tone retention. Hence, the permutation σ must have the property that

$$(\sigma(i)' - \sigma(j)')(k-j)(i-j) + \sigma(j)' = \sigma(k)'. \tag{1}$$

So far, we have shown that the existence and uniqueness of an automorphism Θ that preserves double common-tone retention is equivalent to the existence and uniqueness of σ such that the above equation holds and that Θ is a bijection. We will check single common-tone retention later.

First, we show existence of such a σ. Let σ be the identity permutation. Then the equation reduces to $(i' - j')(k-j)(i-j) + j' = k'$. We shall show that the following equation holds:

$$(i' - j')(k-j)(i-j) = (k' - j').$$

Now since $g^{-1}(\{a, b, c\}) = (0, +)$ and $g'^{-1}(\{a', b', c'\}) = (0, +)$, lemma 3.3 implies that the above equation can be rewritten as

$$(b' - a')(b-c)(b-a) = (b' - c').$$

Once again, recalling that all units in the ring $\mathbb{Z}_{12}$ have the property that $x^{-1} = x$, we can rewrite the equation as

$$(b-a)(b-c) = (b' - a')(b' - c').$$

which is true by duality, proving existence.

For uniqueness, one may pursue analogous arguments as above and show that the other permutations on the letters $\{i, j, k\}$ do not work for equation (1).

Finally, we check that Θ is an automorphism. Since Θ sends the generators W_i and W_j to the generators W'_i and W'_j, it is an isomorphism.

Step 4. We have shown that duality implies the uniqueness and existence of an automorphism that preserves double common-tone retention operators. Now we show that it also preserves single common-tone retention operators. We shall show that $\Theta(W_l) = W_{l'}$, $\Theta(W_m) = W_{m'}$, and $\Theta(W_n) = W_{n'}$.

Note that $W_l = (W_i W_j)^\alpha W_j$, where $\alpha = (l-j)(i-j)$. Then

$$\begin{aligned}\Theta(W_l) &= \Theta\big((W_i W_j)^{(l-j)(i-j)} W_j\big)\\ &= \big((W_{i'} W_{j'})^{(l-j)(i-j)} W_{j'}\big).\end{aligned}$$

Hence, we must now show that

$$(i'-j')(l-j)(i-j) + j' = l'$$

which is equivalent to

$$(i'-j')(l-j)(i-j) = l' - j'.$$

Recalling once more that the units in the ring $\mathbb{Z}_{12}$ have the property that $x^{-1} = x$,

$$(i'-j')(l'-j') = (i-j)(l-j). \tag{2}$$

Now, because $g^{-1}(\{a,b,c\}) = (0,+)$ and $g'^{-1}(\{a',b',c'\}) = (0,+)$, lemma 3.3 implies that this is equivalent to showing

$$(b'-a')(l'-j') = (b-a)(l-j).$$

But because g and g' are Riemannian coordinate systems, it is also true by lemma 4.4, $l = 2a$ and $j = c + a$. Likewise, $l' = 2a'$ and $j' = c' + a'$. Thus, $(l-j) = (2a - (c+a)) = (a-c)$ and $(l'-j') = (2a' - (c'+a')) = (a'-c')$. Now it follows that we must show

$$(b'-a')(a'-c') = (b-a)(a-c) \tag{3}$$

which follows once again by duality.

Showing $\Theta(W_n) = W_{n'}$ is exactly analogous. However one cannot show $\Theta(W_m) = W_{m'}$ in precisely the same manner as the final equation yielded is

$$(b'-a')(2b'-c'+a') = (b-a)(2b-c+a)$$

which does not necessarily follow from duality.

Hence to show $\Theta(W_m) = W_{m'}$, we use the fact that $l' - m' = 2(a'-b')$ by lemma 3.3. Indeed, showing $\Theta(W_m) = W_{m'}$ is equivalent to showing that $(i'-j')(m-j)(i-j)+j' = m'$. But we already know that $(i'-j')(l-j)(i-j)+j' = l'$ because $\Theta(W_l) = W_{l'}$ and so it suffices to show that $l' - m' = (i'-j')(l-j)(i-j) + j' - [(i'-j')(m-j)(i-j)+j'] = 2(a'-b')$. By lemma 3.3, this is equivalent to showing

$$(b'-a')(l-j)(b-a) - (b'-a')(m-j)(b-a) = 2(a'-b').$$

Now,

$$\begin{aligned}(b'-a')(l-j)(b-a) - (b'-a')(m-j)(b-a) &= (b'-a')(b-a)[(l-j)-(m-j)]\\ &= (b'-a')(b-a)(l-m)\\ &= (b'-a')(b-a)(2(a-b))\\ &= (a'-b')(a-b)(2(a-b))\\ &= (a'-b')2(a-b)^2\\ &= 2(a'-b')\end{aligned}$$

□

Corollary 4.5 (Voice-Leading Automorphism on the Riemannian Group)**.** *Let* γ *and* γ' *be dual pc-set classes with coordinate systems* f *and* f'*, respectively. Then* $\exists \Phi : \mathcal{R} \to \mathcal{R}$*, an automorphism, such that*

$$\#_r^* \circ \Phi = \#_r^*.$$

5. Thoughts on Generalizations and Extensions

One possibility for generalization is to consider pc-set classes of chords with more than three notes. One possible generalization of duality is to say that pc-set classes γ and γ' are dual if for all $\{a_0, a_1, \cdots, a_j\} \in \gamma$ and for all $\{a'_0, a'_1, \cdots, a'_j\} \in \gamma'$, $\gcd(a_1 - a_0, 12) = 1$, $\gcd(a'_1 - a'_0, 12) = 1$, and $(a_1 - a_0)(a_i - a_k) = (a'_1 - a'_0)(a'_i - a'_k)$ $\forall i, k$. Since it is unclear what the retention operators would be in this general setting, one might work backwards and investigate precisely what the isomorphisms preserve with this notion of duality.

It is also interesting to consider whether the voice-leading isomorphism can be generalized when the group "upstairs" is no longer $\mathcal{R}$ but rather $\mathcal{R}^*$, the group of skew-Riemannian operators.

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