THE VOICE-LEADING AUTOMORPHISM AND RIEMANNIAN OPERATORS

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Abstract. In the early formulations of Riemannian music theory, the Riemannian operators were defined at least in part by their voice-leading properties. However, some have suggested a root-intervallic approach to the operators. This has the advantage of crystallizing the operators' algebraic properties, but has the disadvantage of abandoning their voice-leading properties. In this paper, we show that there exist classes of set classes for which it is possible to define automorphisms on the Riemannian group that preseve their voice-leading properties.

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This paper proves the existence and uniqueness of an automorphism on the Riemannian group that preserves voice-leading properties. The musical significance of this result is as follows: in Western music, there are two different (and conflicting) ways to think about how to move from one chord to another. The first way is to think of the chords as a single unit, only considering the initial chord and the destination chord. For example, a G-major chord moves down to a C-major chord. This first way of thinking about chord progressions (popular among contemporary guitar players) is called "root-intervallic" motion, and it reflects modern inclinations of thinking about chords as single units. The second way is to think about the individual pitches that comprise the chord, and determine how each pitch in the chord will move. For example, to go from the G-major chord to the C-major chord, perhaps the G in the G-major chord will move down to the C in the C-major chord,

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while the D in the G-major chord will move up to the E in the E-major chord, and so on. This level of detail in pitch-by-pitch specificity is called the "voice-leading" of the chord progression from G-major to C-major.

The two methods of thinking about chord progressions, root-intervallic and voice-leading, present two different ways of thinking about moving from one chord to another. In common-practice Western music, there are idiomatic rules for how both aspects of a chord progression should look like. (As a grossly simplified example, in the key of C-major, an G-major chord is not allowed to move to a F-major chord. This is the root-intervallic constraint. On the other hand, the pitch B in the G-major chord must move to a C. This is the voice-leading constraint.) These two aspects are often at odds with one another, and a solution must be found that satisfies both. Indeed, such problem-solving is the focus of most first-semester music theory courses.

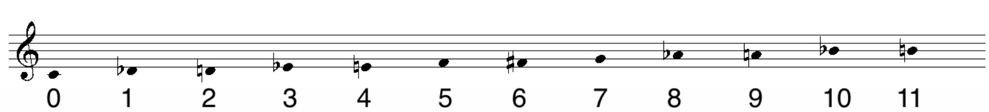
As expected, there is much interesting mathematics that govern successful resolution of such musical conflicts. This paper explores these ideas. We first begin by providing a basic overview of mathematical music theory necessary to understand the rest of this paper.

1. Preliminaries in Mathematical Music Theory

1.1. **Musical Set Theory.** This section is a brief overview of Allen Forte's "musical set theory", which has formed the foundation to all of modern mathematical music theory. Note that "musical set theory" has no relation to "set theory" in mathematics. Also, we warn that the standard nomenclature and terminology in musical set theory is obtuse and sometimes at odds with conventions in mathematics. Nevertheless, these terms are well-adopted, so we stick to them.

Due to the cognitive phenomenon known as *octave equivalence*, we identify two different pitches that are an octave apart to be the same. Since there are 12 notes between an octave in Western music, we make the following definition:

Definition 1.1 (Pitch Class)**.** *An element of $\mathbb{Z}_{12}$ shall be identified with the 12 notes between an octave. We arbitrary designate each integer to a pitch as shown below:*



Each element of $\mathbb{Z}_{12}$ shall be called a pitch class, sometimes abbreviated as pc.

In music, we are often concerned with pitches sounding simultaneously:

Definition 1.2 (pc-Set)**.** *A pc-set is any subset of $\mathbb{Z}_{12}$. In other words, α is a pc-set if $\alpha \in \mathcal{P}(\mathbb{Z}_{12})$, the power set of $\mathbb{Z}_{12}$.*

We understand that the pc-sets denote pitches sounding simultaneously or in close temporal proximity to one another as to form a single musical event (musicians may refer to these two interpretations as *harmony* and *melody*, respectively.)

Now we introduce a useful way to measure the distance between two pitch classes.

Definition 1.3 (pc Interval Class)**.** *Let $a, b \in \mathbb{Z}_{12}$. Then the interval class between a and b, denoted $d(a, b)$, is* $\min\{(a-b) \mod 12, (b-a) \mod 12\}$.

If $d(a, b) = d(c, d)$, then the pc-sets $\{a, b\}$ and $\{c, d\}$ will have similar enough sonic quality to be considered equivalent. Musicians may refer to this common character between $\{a, b\}$ and $\{c, d\}$ as having the same "sound" or "harmony". We seek to generalize this notion, so that all pc-sets with the same sonic quality may be identified as equivalent. This provides a mathematical rigorous way of defining what we mean by "harmony". In order to do this, we need a way to take inventory of all interval classes contained within a pc-set. A mathematico-musical object known as the *interval vector* does precisely this. For clarity and demonstration, we first compute the interval vector before defining it:

Consider the pc-set $\{0, 4, 5, 6\}$. First, we compute the interval-class between every possible pair of pc's in the pc-set:

$$d(0,4) = 4$$
$$d(0,5) = 5$$
$$d(0,6) = 6$$
$$d(4,5) = 1$$
$$d(4,6) = 2$$
$$d(5,6) = 1$$

We may summarize the above computation with the following table:

Interval Class	1	2	3	4	5	6
# of times it appears	2	1	0	1	1	1

Notice that the above table enumerates every "sound" or interval class that is heard in the pc-set when it is played. We can write the table even more succinctly by eliminating the first row, and simply writing:

$$[2, 1, 0, 1, 1, 1].$$

The above 6-tuple is called the *interval vector* of $\{0, 4, 5, 6\}$.

Definition 1.4 (Interval Vector). *Let α be a pc-set. Let i_n be the number of times the interval class n appears beween pc's in α. Then $[i_1, i_2, i_3, i_4, i_5, i_6]$ is called the interval vector of α.*

Thus, if two pc-sets share the same interval vector, they will "sound" the same because both have the same "content" in terms of the distance between sounding pitches. For example, all major triads have the same interval vector, as do all minor triads, or dominant-seventh chords, and so on. When a musician says something like "Use the augmented dominant ninth chord", they are singling out a particular interval vector.

Now we introduce two operators on the space of all pc-sets that are musically significant:

Definition 1.5 (The Transposition Operator). *Let $T_n : \mathcal{P}(\mathbb{Z}_{12}) \to \mathcal{P}(\mathbb{Z}_{12})$, where if $\{a_1, a_2, \cdots, a_i\} \in \mathcal{P}(\mathbb{Z}_{12})$, then $T_n(\{a_1, a_2, \cdots, a_i\}) = \{a_1+n, a_2+n, \cdots, a_i+n\}$, where all addition takes place in $\mathbb{Z}_{12}$.*

The term "transposition" has nothing to do with transpositions in permutations.

As an example, consider the pc-class $M = \{0, 4, 7\}$, which is called a C major chord in Western music. Then $T_7(M) = \{7, 11, 2\}$, which is a G major chord.

Definition 1.6 (The Inversion Operator)**.** *Let* $I : \mathcal{P}(\mathbb{Z}_{12}) \to \mathcal{P}(\mathbb{Z}_{12})$*, where if*$\{a_1, a_2, \cdots, a_i\} \in \mathcal{P}(\mathbb{Z}_{12})$*, then* $I(\{a_1, a_2, \cdots, a_i\}) = \{-a_1, -a_2, \cdots, -a_i\}$*, where the negation takes place in* $\mathbb{Z}_{12}$*.*

For example, consider the pc-class $M = \{0, 4, 7\}$, which is called a C major chord in Western music. Then $I(M) = \{0, 8, 5\}$, which is a f minor chord.

The following theorem shows the significance of the above two definitions:

Theorem 1.7. *Let* α *be a pc-set.*

(1) α *and* $T_n(\alpha)$ *have the same interval vector.*
(2) α *and* $I(\alpha)$ *have the same interval vector.*

We omit the proof.

Thus, T_n and I provide a way of generating new pc-sets that share the same interval vector. Indeed, musicians have been using transpositions and inversions to generate chords that sound the "same" since the middle ages. Thus, we arrive at the following definition:

Definition 1.8 (Set Class)**.** *Fix some* m*. Let* $\mathcal{P}_m(\mathbb{Z}_{12})$ *denote the set of all pc-sets of cardinality* m*. For all* $U, V \in \mathcal{P}_m(\mathbb{Z}_{12})$*, let* $U \sim V$ *if either one of the following hold:*

(1) $U = T_n(V)$ *for some* n
(2) $U = T_n \circ I(V)$ *for some* n

Then $\sim$ *is an equivalence relation on* $\mathcal{P}_m(\mathbb{Z}_{12})$*. We call the equivalence classes the set classes of order* m*.*

For example, the set of all major and minor triads form a set class of order 3, as one can verify. As a more complicated example, the set of all dominant seventh chords and half-diminished seventh chords form a set class of order 4.

An immediate consequence of 1.7 and 1.8 is the following corollary:

Corollary 1.9. *Let* α *and* β *be pc-sets. If* α *and* β *belong to the same set class, then* α *and* β *have the same interval vector.*

Interestingly, the converse does not hold. If distinct set classes γ and γ' have the same interval vector, they are called Z-related. Providing necessary and sufficient conditions of Z-related set classes is a still on-going area of research.

Despite the above set back (the lack of the converse), set classes still provide the most concrete characterization of what musicians mean when they say two chords are the "same". Even for Z-related set classes, despite sharing the same interval vector and hence the same sonority, the lack of a transposition or inversion operator that maps one to another makes its musical application limited.

We make one final remark about the cardinality of set classes. Let γ be a set class of order n. Then, at most, $|\gamma| = 24$. Suppose $|\gamma| = 24$. Then we define an additional equivalence relation within γ: let $U \sim_t V$ iff $U = T_n(V)$. This will yield two equivalence classes, which we shall denote γ^+ and γ^-. These are called *transposition classes*, and $I(\gamma^+) = \gamma^-$.

1.2. **Neo-Riemannian Music Theory.** Our discussion so-far has focused on individual chords, or set classes, in isolation. But in music, these chords are juxtaposed to one another temporally, forming what musicians call *chord progressions*: one pc-set is heard, then another, then another, and so on. The rigorous study of such processes is called *transformation theory* or neo-riemannian theory, after the music theorist Hugo Riemann.

While a detailed exposition of this area of research is beyond the scope of this preliminary section, only a few definitions are logically necessary for the rest of this paper.

We motivate the ensuing discussion with an example. Consider the set-class γ of major and minor triads:

$$\begin{array}{c}
\gamma^+\gamma^- \\
\{0,4,7\}\{0,3,7\} \\
\{1,5,8\}\{1,4,8\} \\
\{2,6,9\}\{2,5,9\} \\
\{3,7,10\}\{3,6,10\} \\
\{4,8,11\}\{4,7,11\} \\
\{5,9,0\}\{5,8,0\} \\
\{6,10,1\}\{6,9,1\} \\
\{7,11,2\}\{7,10,2\} \\
\{8,0,3\}\{8,11,3\} \\
\{9,1,4\}\{9,0,4\} \\
\{10,2,5\}\{10,1,5\} \\
\{11,3,6\}\{11,2,6\}
\end{array}$$

In music, we call the pc-set $\{0,4,7\}$ C-major. Notice that of the three pitches in $\{0,4,7\}$, one element, 0 (or C), has been singled-out to label the the pc-set as "C-major". Furthermore, the term "major" refers to the transposition class $\{0,4,7\}$ belongs to. In other words, we just need 2 parameters to determine an element of a set class: a canonical representative (such as 0 in $\{0,4,7\}$), and the transposition class (such as "major"). (Musicians will refer to the canonical representative as the "root" of the pc-set, and the transposition class as the "mode"). This motivates the following definition:

Definition 1.10 (Coordinate Space)**.** *Let γ be a set class with 24 pc-sets. Let $\Gamma = \mathbb{Z}_{12} \times \{+,-\}$. When Γ has been identified with γ (subject to rules defined in the next section), Γ is called the coordinate space of γ.*

Indeed, some music theorists in transformation theory simply choose to work with the coordinate space Γ, ignoring the underlying set class γ. For example, this is the approach Hook takes in [2].

We need only make one more crucial definition before leaving this preliminary section. While this definition may seem unmotivated, its significance will be explained shortly:

Definition 1.11 (The Riemannian Group)**.** *Consider some coordinate space Γ.*

(1) *Let $S_n : \Gamma \to \Gamma$, where* $S_n\big[(r,\Delta)\big] = \begin{cases} (r+n,\Delta) \text{ if } \Delta = + \\ (r-n,\Delta) \text{ if } \Delta = - \end{cases}$

(2) *Let* $W_n : \Gamma \to \Gamma$*, where* $W_n\big[(r,\Delta)\big] = \begin{cases} (r+n, -\Delta) \text{ if } \Delta = + \\ (r-n, -\Delta) \text{ if } \Delta = - \end{cases}$

Then the set of all S_n *and* W_n *for* $n \in \mathbb{Z}_{12}$ *form a group with function composition as the group operation. This group, denoted* $\mathcal{R}$*, is called the Riemannian group and it is isomorphic to* $\mathcal{D}_{12}$*, the dihedral group of a 12-gon.*

While we omit a formal proof, one can see $\mathcal{R}$ as $\mathcal{D}_{12}$ by thinking of S_n as rotations and W_n as reflections.

The above group has gradually (and somewhat tortuously) been agreed upon in the literature as the groups of transformations on Γ that is the most musically relevant. While it is difficult to provide a flavor for its significance here, we shall provide a short musical and mathematical justification. Musically, $\mathcal{R}$ is important because much of common chord progressions in Western music form important subgroups of $\mathcal{R}$. For example, there are sections of Bach violin concertos that form a cyclic subgroup of $\mathcal{R}$.

For its mathematical significance, we cite the following list of properties, courtesy [2]:

Theorem 1.12 (The Mathematical Significance of $\mathcal{R}$)**.** *The Riemannian group has the following properties:*

1. $\mathcal{R}$ *is simply transitive on* Γ
2. $\mathcal{R}$ *is a normal subgroup of a larger class of transformations on* Γ*, called the Uniform Triadic Transformations (UTT for short).*
3. $\mathcal{R}$ *is generated by* W_4 *and* W_9*, each of which are musically significant. (Both* W_4 *and* W_9*, when applied to the set class of major and minor chords, have an important musical property called "common-tone retention". This will be explored in the following section).*

As a vague but tantalizing side-note, the Riemannian group $\mathcal{R}$, when applied to the set class of major and minor triads, generates a surface that has the topology of a torus.

This concludes the preliminary section of this paper. We hope that the above discussion, even thought terse and incomplete, provides enough background and motivation to understand the rest of this paper.

Now we first provide a brief summary the material from here onwards. We recommend that the reader skip directly to the next section on first reading, and return to this part for the big-picture discussion later.

In [2], Hook defines Riemannian Operators using root-interval motion. This has the advantage of the operators being easily generalizable to arbitrary set classes. However, the voice-leading properties of the operators, such as common-tone retention, are lost when not working with major and minor triads. For example, the *leittonwechsel* operator, when considered as a uniform triadic transformation acting on the set class 3-11, becomes W_4. But, W_4 does not have the hallmark double-common tone retention voice-leading property when applied in other set classes. (For example, see [3])

However, one could argue that the double-common tone retention of the *leittonwechsel* is almost a defining characteristic of that operator, as originally conceived by Riemann and later others (such as [1] and [5]). Hence, it would seem that while the *leittonwechsel* coincides with W_4 on the set class 3-11, perhaps it should coincide with other operators on other set classes. Indeed, the root-interval motion of the leittonwechsel operator on the set class 3-11 appears to be its least interesting aspect. The most interesting aspect is its voice-leading properties and its preservation of the intervals.

Put another way, the voice-leading properties of the Riemannian operators, when defined root-intervallically, are not inherent in themselves. For example, the common-tone retention properties of the Riemannian operators change depending on which set class the operators act on. But is there some pattern or structure to the way that this property changes when moving from one set class to another?

In this paper, we show that if we move from one set class to another, there exists an automorphism on the Riemannian group that preserves the voice-leading properties of the operators, as long as the two set classes involved are "dual" to each other in a way that will be defined later. Along the way, we will also provide a definitive proof to the notion that the root-intervallic approach and the voice-leading approach to Riemannian operators are equivalent in some way.

In order to make this concept precise and rigorous, it will be necessary to deal simultaneously with operators that are defined root-intervallicaly (as in [2]) and operators that are defined according to their common-tone retention properties (as in [1]). Hence, for the purposes of this paper, the term *Riemannian Group* shall refer to the subgroup of the uniform triadic transformations as defined by Hook, and we shall use the term *Retention Group* for the group that is generated by double common-tone retention operators.

Indeed, much of the literature in Riemannian theory seem to conflate these two groups. The Riemannian group and the retention group are often considered different approaches to the same object. In fact they are mathematically different objects, although we shall prove in section 3 that they are, in fact, isomorphic. But the situation is more subtle— the isomorphism is not canonical. This means that the voice-leading properties of the Riemannian operators change depending on the underlying set class, as already stated.

In [2], the retention group in the set class of major and minor triads are taken as motivation for defining the Riemannian group. This paper will show how the Riemannian group relates to the retention group in other set classes.

In sections 2 and 3, we provide a rigorous exposition of the already-known fact that different choices in the root or the canonical representative in an arbitrary set class correspond to a "relabeling" of the Riemannian operators. But more importantly, this section introduces key notation used in sections 4 and 5, and also presents a theorem that simplifies the proof of the main result in section 5.

In section 5, we define the notion of dual set classes of trichords, and show that the voice-leading properties of the Riemannian operators are algebraically compatible(that is, an automorphism exists that preserves common-tone retention) if and only if the underlying set classes are dual. Here is the statement of the main result, given now for immediacy.

Theorem 1.13 (Voice-Leading Automorphism on the Riemannian Group). *Let γ and γ' be dual set classes with coordinate systems f and f', respectively. Then $\exists! \Phi : \mathcal{R} \to \mathcal{R}$, an automorphism, such that if the operator $U \in \mathcal{R}$ has n common-tone retention with respect to f on γ, then $\Phi(U)$ also has n common-tone retention with respect to the coordinate system f' on γ'*

In section 5, we attempt to generalize this result to tetrachordal set classes. We provide a numerical verification that suggests that the voice-leading automorphism is unique for tetrachords as well.

2. Coordinate Systems on set classes

Throughout this section, $\Gamma = \{(r, \Delta) \mid r \in \mathbb{Z}_{12},\ \Delta \in \{+, -\}\}$. That is, Γ is the space used to represent major and minor triads as in [2]. $\mathcal{R}$ is the group of Riemannian operators on Γ. Also, γ shall denote any set class. If γ has two transposition classes, we write them as γ^+ and γ^- so that $\gamma^+ \cup \gamma^- = \gamma$. Also, we summarize the following basic definitions to fix notation:

Definition 2.1 (Transposition Operators)**.**

(1) *Let* $T_n : \gamma \to \gamma$, *where* $T_n\Big[\{a_0, a_1, a_2, \cdots, a_j\}\Big] = \{a_0 + n, a_1 + n, a_2 + n, \cdots, a_j + n\}$.
(2) *Let* $T_n^* : \Gamma \to \Gamma$, *where* $T_n^*\Big[(r, \Delta)\Big] = (r + n, \Delta)$.

Definition 2.2 (Inversion Operators)**.**

(1) *Let* $I : \gamma \to \gamma$, *where* $I\Big[\{a_0, a_1, a_2, \cdots, a_j\}\Big] = \{-a_0, -a_1, -a_2, \cdots, -a_j\}$.
(2) *Let* $I^* : \Gamma \to \Gamma$, *where* $I^*\Big[(r, \Delta)\Big] = (r, -\Delta)$.

Clearly, the elements of Γ can be used to represent the elements of γ by defining a bijection between the two sets. But not all identifications between Γ and γ are musically significant. Hence, we shall now define a particular type of bijection called a *coordinate system.* While the following definition may seem very general and elegant, we shall soon see that the coordinate systems are precisely the types of representations that are musically significant. In particular, we may think of them as a choice of tonic in a rough sense.

Definition 2.3 (Coordinate System)**.** *Let* $\Gamma = \{(r, \Delta) \mid r \in \mathbb{Z}_{12}, \Delta \in \{+, -\}\}$. *Let* γ *be a set class with* 24 *elements. A coordinate system on* γ *is a bijection* $f : \Gamma \to \gamma$ *such that* $\forall n$,

$$T_n \circ f = f \circ T_n^*.$$

As a passing note, we mention that the above definition bares a striking resemblance to that of the uniform triadic transformations. More specifically, the UTT group can be shown to be the centralizer of the transposition operators.

Now we shall prove that the above definition has very musical properties.

Theorem 2.4. *Suppose that* f *is a coordinate system. Then*

(1) *For any* $r \in \mathbb{Z}_{12}$, $I \circ f\big[(r, \Delta)\big] = f\big[(s, -\Delta)\big]$ *for some* $s \in \mathbb{Z}_{12}$.
(2) *For any* $r \in \mathbb{Z}_{12}$, $T_n \circ f\big[(r, \Delta)\big] = f\big[(r + n, \Delta)\big]$.

Roughly speaking, the above theorem shows that if γ^+ and γ^- are the two transposition classes of a set class, then a coordinate system assigns all of γ^+ to a tuple of the form (r, Δ) and all of γ^- to a tuple of the form $(r, -\Delta)$. Furthermore, a transposition in Γ is precisely the same as the corresponding transposition in γ.

Proof. (2) follows trivially from the definition of coordinate systems.

To show (1), note that because γ is a set class with 24 elements, $I \circ f\big[(r, \Delta)\big] \neq T_n \circ f\big[(r, \Delta)\big]$ for any n. But $T_n \circ f\big[(r, \Delta)\big] = f \circ T_n^*\big[(r, \Delta)\big]$, so $I \circ f\big[(r, \Delta)\big] \neq$

$f \circ T_n^*\big[(r,\Delta)\big]$. In other words, $I \circ f\big[(r,\Delta)\big] \neq f\big[(n,\Delta)\big]\ \forall n$. Hence, $\exists s$ s.t. $I \circ f\big[(r,\Delta)\big] = f\big[(s,-\Delta)\big]$.

□

We note that choosing a coordinate system for a set class is a notion that generalizes picking a "root" for chords in a set class. For example, representing the major and minor triads as an ordered tuple (r,Δ) where r is the root of the triad and Δ is $+$ or $-$ depending on whether the triad is major or minor, respectively, is a particular coordinate system on the set class 3-11. Or, suppose we represent the major and minor triads as above with the exception that for minor triads, we pick what is traditionally the "fifth" as the root, so that, for example, $(0,-)$ is f-minor. This is the traditional Riemannian scheme, and will be of some importance in section 4.

A different approach to picking "canonical representatives" or "roots" have been explored in the literature before (see [6] and [4]). It is interesting that such a musically intuitive concept is difficult to pin down mathematically. I believe that the definition given in this paper is the most general, and will prove to be the most useful for the rest of this paper.

Note that choosing a coordinate system also may correspond to picking the "tonic". So for example, the coordinate system given above for major/minor triads may correspond to the key of C-Major, but the coordinate system that assigns the g-minor triad to $(0,+)$ and the G-Major triad to $(0,-)$ may correspond to the key of g-minor.

3. Preservation Automorphisms

Now, depending on the choice of a coordinate system, the action of a Riemannian operator on the underlying set class space changes, as the Riemannian operators are defined root-intervallicaly. The following important theorem shows that the action of Riemannian operators can be preserved under a change of coordinate system via an automorphism:

Theorem 3.1 (Preservation Automorphism)**.** *Let f and g be coordinate systems on a set class γ. Then there exists an automorphism Ω on $\mathcal{R}$ such that for all $U \in \mathcal{R}$,*

$$f \circ U \circ f^{-1} = g \circ \Omega(U) \circ g^{-1}.$$

In other words, the following diagram commutes:

$$\begin{array}{ccccccc} \gamma & \xrightarrow{f^{-1}} & \Gamma & \xrightarrow{U} & \Gamma & \xrightarrow{f} & \gamma \\ & \overset{g^{-1}}{\searrow} & \Gamma & \xrightarrow{\Omega(U)} & \Gamma & \overset{g}{\nearrow} & \end{array}$$

We shall call Ω the preservation automorphism corresponding to the ordered pair of coordinate transformations (f,g).

We shall delay the proof of this theorem until the preservation automorphisms are given explicitly in a later theorem. In order to give the automorphism explicitly, we need to define how to precisely measure the "difference" between two coordinate systems.

Definition 3.2 (Modulation Function)**.** *Let $\mathcal{C}$ be the set of all coordinate systems on γ. Let $f,g \in \mathcal{C}$. The modulation function $D : \mathcal{C} \times \mathcal{C} \to \Gamma$ is given by $D(f,g) = (r,\sigma)$ where*

(1) (a) $\sigma = +$ *if* $I \circ g\big[(x, \Delta)\big] = f\big[(s, -\Delta)\big]$ *for some* s
(b) $\sigma = -$ *if* $I \circ g\big[(x, \Delta)\big] = f\big[(s, \Delta)\big]$ *for some* s

(2) $r = j - k$ *where* $\forall x \in \mathbb{Z}_{12}$,
$g\big[(x, \sigma)\big] = T_j \circ f\big[(x, +)\big]$ *and*
$g\big[(x, -\sigma)\big] = T_k \circ f\big[(x, -)\big]$.

Now we define some specific automorphisms on $\mathcal{R}$ that will be useful later. Recall that the Riemannian group $\mathcal{R}$ is generated by the elements S_1 and W_0, where $S_n = (S_1)^n$ and $W_n = (S_1)^n W_0$. Hence, an endomorphism on $\mathcal{R}$ may be fully characterized by its action on these generators.

Theorem 3.3 (Automophisms on $\mathcal{R}$)**.** *Let* $r \in \mathbb{Z}_{12}$. *Let* $\phi^{S_{r/2}}$ *be the endomorphism where* $S_1 \mapsto S_1$ *and* $W_0 \mapsto W_r$. *Also let* $\phi^{W_{r/2}}$ *be the endomorphism where* $S_1 \mapsto S_{11}$ *and* $W_0 \mapsto W_r$. *Then* $\phi^{S_{r/2}}$ *and* $\phi^{W_{r/2}}$ *are automorphisms on* $\mathcal{R}$. *Furthermore, they are inner automorphisms if and only if* r *is even.*

Theorem 3.4 (Preservation Automorphisms)**.** *Let* f *and* g *be coordinate transformations on a set class* γ. *Then then preservation automorphism* Ω *corresponding to* (f, g) *is given by the following:*

(1) *If* $D(f, g) = (r, +)$, *then* $\Omega = \phi^{S_{r/2}}$.
(2) *If* $D(f, g) = (r, -)$, *then* $\Omega = \phi^{W_{r/2}}$.

Proof. Fix some $U \in \mathcal{R}$. Then $U = W_n$ or S_n. Furthermore, suppose that $D(f, g) = (r, +)$. We shall show that $f \circ U \circ f^{-1} = g \circ \Omega(U) \circ g^{-1}$ by direct computation. So first suppose that $U = S_n$. We proceed by computing on a typical element of γ:

$$\begin{aligned}
g \circ \Omega(S_n) \circ g^{-1}(\{a,b,c\}) &= g \circ \phi^{S_{r/2}}(S_1^n) \circ g^{-1}(\{a,b,c\}) \\
&= g \circ \phi^{S_{r/2}}(S_1)^n \circ g^{-1}(\{a,b,c\}) \\
&= g \circ S_1^n \circ g^{-1}(\{a,b,c\}) \\
&= g \circ S_n \circ g^{-1}(\{a,b,c\}) \\
&= \begin{cases} T_j \circ f \circ S_n \circ g^{-1}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = + \\ T_k \circ f \circ S_n \circ g^{-1}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = - \end{cases} \\
&= \begin{cases} T_j \circ f \circ S_n \circ f^{-1} \circ T_{-j}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = + \\ T_k \circ f \circ S_n \circ g^{-1} \circ T_{-k}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = - \end{cases} \\
&= \begin{cases} f \circ T_j^* \circ S_n \circ T_{-j}^* \circ f^{-1}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = + \\ f \circ T_k^* \circ S_n \circ T_{-k}^* \circ g^{-1}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = - \end{cases} \\
&= \begin{cases} f \circ S_{j+n-j} \circ f^{-1}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = + \\ f \circ S_{-k+n+k} \circ g^{-1}(\{a,b,c\}) & \text{if } \pi_2\big(g^{-1}(\{a,b,c\})\big) = - \end{cases} \\
&= f \circ S_n \circ f^{-1}(\{a,b,c\})
\end{aligned}$$

The computations for $U = W_n$ and $D(f, g) = (r, -)$ are very analogous. □

4. Retention Operators

For this section and the next, γ shall always denote set classes of trichords.

Note that once a specific coordinate system for a set class has been chosen, a Riemannian operator induces a function on the set class directly:

$$\begin{array}{ccc} \Gamma & \xrightarrow{U} & \Gamma \\ f\downarrow & & f\downarrow \\ \gamma & \cdots> & \gamma \end{array}$$

We can define the group of operators acting on γ with function composition as the group operation. We shall denote this group by $\mathcal{C}$, relative to the set class γ.

We shall see shortly that the group $\mathcal{C}$ "downstairs" are precisely those generated by the double-common tone retention operators. Hence, we clarify some terminology first. If an operator $U : \gamma \to \gamma$ has the property that for all $a \in \gamma$, $|a \cap G(a)| = n$ and $n \neq 0$, then we shall call G a retention operator and the elements of $a \cap G(a)$ the *retention notes* of G. Furthermore, n shall be called the retention value of G, denoted by $\#_r(G)$. We clarify the above discussion with the following definition:

Definition 4.1 (Retention Value)**.** *Let γ be a set class with a coordinate system f.*

(1) *Suppose that $G \in \mathcal{C}$ has the property that for all $a \in \gamma$, $|a \cap G(a)| = n$. Then the retention value of G is $\#_r(G) = n$.*
(2) *The retention value of $U \in \mathcal{R}$ relative to f is $\#_r^* = \#_r(f \circ U \circ f^{-1})$.*

Hook comments that the voice-leading properties of the Riemannian operators (that is, the retention values of the operators) change when moving from one set class to another. However, we shall soon see that we can define the notion of dual set classes, and that the retention properties of the Riemannian operators can be preserved via an automorphism if and only if one moves from a set class to its dual.

To illustrate this, consider an operator G on the set class γ induced by a Riemannian operator U. Now apply the same operator G with the same voice-leading properties on γ'. This, in turn, induces a Riemannian operator U' as the following diagram demonstrates:

$$\begin{array}{ccc} \Gamma & \xrightarrow{U} & \Gamma \\ f\downarrow & & f\downarrow \\ \gamma & \overset{\cdots>}{G} & \gamma \end{array} \Rightarrow \begin{array}{ccc} \Gamma & \overset{U'}{\cdots>} & \Gamma \\ g\downarrow & & g\downarrow \\ \gamma' & \overset{\longrightarrow}{G} & \gamma' \end{array}$$

It turns out that this mapping from U to U' is an automorphism on $\mathcal{R}$ if and only if the underlying set classes are dual to each other. Note that in the special case where $\gamma = \gamma'$, the above situation is precisely a change of coordinate systems.

But first, we shall work to show that $\mathcal{C}$ is precisely the group generated by the operators with common-tone retention.

Definition 4.2 (Retention Operators)**.** *Fix some coordinate system f on γ. Also fix some chord $\{a, b, c\} \in \gamma$. (Sometimes, it is convenient to think of $\{a,b,c\}$ as the prime form of the set class, but this is not a logical necessity.)*

(1) *The retention operator $R_{a,b} : \gamma \to \gamma$ shall be the operator $f \circ U \circ f^{-1}$ where U is the unique Riemannian operator $U : \Gamma \to \Gamma$ such that $f \circ U \circ f^{-1}(\{a, b, c\}) = \{a, b, x\}$ where $x \neq c$.*
(2) *The retention operator $R_a : \gamma \to \gamma$ shall be the operator $f \circ W_n \circ f^{-1}$ where W_n is the unique Weschel operator $W_n : \Gamma \to \Gamma$ such that $f \circ W_n \circ f^{-1}(\{a, b, c\}) = \{a, x, y\}$ where $x, y \neq b, c$.*

For convenience, we shall sometimes write $R_{a,b} \sim U$ to mean $R_{a,b} = f \circ U \circ f^{-1}$ and $R_a \sim W_n$ to mean $R_a = f \circ W_n \circ f^{-1}$.

This definition perhaps best captures the interplay between the voice-leading properties and its Riemannian dualism of retention operators.

Also note that the above definition slightly abuses notation, as the operator $R_{a,b}$ has as its retention notes $\{a, b\}$ if and only if it acts on the chord {a,b,c}. However, $R_{a,b}$ always has two retention notes as the Riemannian operators are uniform (See [2]) .

We begin with a technical lemma that will be the workhorse of the rest of this paper.

Lemma 4.3. *Suppose that γ is a set class with a coordinate system f. Let $\{a, b, c\} \in \gamma$. Suppose $f^{-1}(\{a, b, c\}) = (r, +)$ for some r. Then:*

(1) *Let $R_{b,c} \sim W_i$, $R_{a,c} \sim W_j$, and $R_{a,b} \sim W_k$. Then*

$$\begin{aligned} i - j &= b - a \\ j - k &= c - b \\ k - i &= a - c \end{aligned}$$

(2) *Also let $R_a \sim W_l$, $R_b \sim W_m$, and $R_c \sim W_n$. Then*

$$\begin{aligned} l - m &= 2(a - b) \\ m - n &= 2(b - c) \\ n - l &= 2(c - a) \end{aligned}$$

If $f^{-1}(\{a, b, c\}) = (r, -)$, then simply add a factor of -1 to one side of each equation given above.

As an interesting consequence of the above lemma, it is always true that $\{i, j, k\} \in \gamma$.

Proof. Part 1. We shall show that $i - j = b - a$ if $f^{-1}(\{a, b, c\}) = (r, +)$ and $i - j = -(b - a)$ if $f^{-1}(\{a, b, c\}) = (r, -)$. Then by symmetry, the other equations in (1) also follow by precisely the analogous arguments.

Note that $R_{b,c}(\{a, b, c\}) = \{b, c, c + b - a\}$ and $R_{a,c}(\{a, b, c\}) = \{a, c, c - b + a\}$ simply by the interval structure of γ. Hence, $f \circ W_i \circ f^{-1}(\{a, b, c\}) = \{b, c, c + b - a\}$ and $f \circ W_j \circ f^{-1}(\{a, b, c\}) = \{a, c, c - b + a\}$. Now, consider the case where $f^{-1}(\{a, b, c\}) = (r, +)$ for some r. Since $T^*_{i-j} \circ W_j[(r, +)] = W_i[(r, +)]$,

$$\begin{aligned} f \circ W_i \circ f^{-1}(\{a, b, c\}) &= f \circ T^*_{i-j} \circ W_j \circ f^{-1}(\{a, b, c\}) \\ &= T_{i-j} \circ f \circ W_j \circ f^{-1}(\{a, b, c\}). \end{aligned}$$

Hence,

$$\begin{aligned} \{b, c, c + b - a\} &= T_{i-j}(\{a, c, c - b + a\}) \\ &= (\{a + i - j, c + i - j, c - b + a + i - j\}. \end{aligned}$$

Now, considering every bijection possible between $\{b, c, c + b - a\}$ and $\{a + i - j, c + i - j, c - b + a + i - j\}$, the only system of equations that does not lead to a contradiction is the one that yields $i - j = b - a$. A similar computation for the case where $f^{-1}(\{a, b, c\}) = (r, -)$ yields $i - j = -(b - a)$.

Part 2. We shall show that $l - m = 2(a - b)$. Then by symmetry, the other equations in (2) also follow by precisely the analogous arguments.

Note that $R_a(\{a, b, c\}) = \{2a - c, 2a - b, a\}$ and $R_b(\{a, b, c\}) = \{b, 2b - a, 2b - c\}$ simply by the interval structure of γ. Hence, $f \circ W_l \circ f^{-1}(\{a, b, c\}) = \{2a - c, 2a - b, a\}$

and $f \circ W_m \circ f^{-1}(\{a,b,c\}) = \{b, 2b-a, 2b-c\}$. Now, consider the case where $f^{-1}(\{a,b,c\}) = (r,+)$ for some r. Since $T^*_{l-m} \circ W_m[(r,+)] = W_l[(r,+)]$,

$$\begin{aligned} f \circ W_i \circ f^{-1}(\{a,b,c\}) &= f \circ T^*_{l-m} \circ W_m \circ f^{-1}(\{a,b,c\}) \\ &= T_{l-m} \circ f \circ W_m \circ f^{-1}(\{a,b,c\}). \end{aligned}$$

Hence,

$$\begin{aligned} \{2a-c, 2a-b, a\} &= T_{l-m}(\{b, 2b-a, 2b-c\}) \\ &= (\{b+l-m, 2b-a+l-m, 2b-c+l-m\}. \end{aligned}$$

Now, considering every bijection possible between $\{2a-c, 2a-b, a\}$ and $\{b+l-m, 2b-a+l-m, 2b-c+l-m\}$, the only system of equations that does not lead to a contradiction is the one that yields $l-m = 2(a-b)$. A similar computation for the case where $f^{-1}(\{a,b,c\}) = (r,-)$ yields $l-m = -2(a-b)$. □

Finally, we arrive at the theorem that shows that the space of operators "downstairs" is isomorphic to the Riemannian group.

Theorem 4.4 (Group Structure of the Voice-Leading Group)**.** *Consider any* $\{a,b,c\} \in \gamma$*. Suppose that* $\gcd(b-a, 12) = 1$*. Then* $R_{b,c}$ *and* $R_{a,c}$ *generates* D_{12}*, the dihedral group on a 12-gon, and hence is isomorphic to* $\mathcal{R}$*. We shall denote this group by* $\mathcal{C}$*.*

This theorem definitively provides conditions for when the voice-leading approach and root-intervallic approach to transformation theory are equivalent in general set classes up to isomorphism. However, it should be noted that this isomorphism is not canonical and certainly does not preserve voice-leading properties in general. We shall have more to say on that matter in the next section.

We note that the only set class of trichords with 24 chords that does not obey the hypotheses of this theorem is the set class 3-8.

Proof. This proof shall proceed by "lifting" upstairs to Γ and computing there. Fix some coordinate system f on γ. Then because γ has 24 chords, there exists $W_i, W_j \in \mathcal{R}$ such that $R_{b,c} = f^{-1} \circ W_i \circ f$ and $R_{a,c} = f^{-1} \circ W_j \circ f$. Because there is a natural isomorphism between the group generated by $\{W_i, W_j\}$ and $\{f^{-1} \circ W_i \circ f, f^{-1} \circ W_j \circ f\}$, $R_{b,c}$ and $R_{a,c}$ generate D_{12} if and only if W_i and W_j does.

But $i - j = \pm(b-a)$ by 4.3. Hence, because $\gcd(b-a, 12) = 1$ by assumption, it also follows now that $\gcd(i-j, 12) = 1$. This implies that W_i and W_j generate D_{12}. □

Corollary 4.5. *Suppose that* γ *follows the hypotheses of the previous theorem. Then the function* $\psi : \mathcal{R} \to \mathcal{C}$ *where* $\psi(U) = G$ *if* $G \sim U$ *(that is,* $G = f \circ U \circ f^{-1}$*) is a group isomorphism.*

5. The Voice-Leading Automorphism and Dual set classes

Now we define what we mean by a dual set class. Later on, we shall see that these dual set classes are precisely the ones with algebraically-compatible voice-leading structures.

Definition 5.1 (Dual set class)**.** *Let* γ *and* γ' *be set classes of trichords. Further suppose that for any* $\{a,b,c\} \in \gamma$ *and* $\{a',b',c'\} \in \gamma'$*,* $\gcd(b-a, 12) = 1$ *and* $\gcd(b'-a', 12) = 1$*. Then* γ *and* γ' *are dual if*

$$(b-a)(b-c) = (b'-a')(b'-c')$$

and

$$(b-a)(c-a) = (b'-a')(c'-a').$$

The following table shows which set classes are dual to which set classes, in prime form:

set class	dual set class
{0,1,3}	{0,2,5}
{0,1,4}	{0,3,7}
{0,1,5}	{0,1,5}
{0,1,6}	{0,1,6}
{0,2,6}	None

TABLE 1. Dual set classes

As an example, we shall compute and see why $\{0,1,3\}$ is dual to $\{0,2,5\}$. Let $a=0, b=1$, and $c=3$, while $a'=5, b'=0$, and $c'=2$. Then, $\gcd(b-a,12) = gcd(1,12) = 1$, while $\gcd(b'-a',12) = \gcd(7,12) = 1$, so that the first part of the definition is satisfied. Now, $(b-a)(b-c) = (1-0)(10-0) = (1)(10) = 10$, while $(b'-a')(b'-c') = (0-5)(0-2) = (7)(10) = 10$. Thus, $(b-a)(b-c) = (b'-a')(b'-c') = 10$. Also, $(b-a)(c-a) = (1)(3) = 3$, while $(b'-a')(c'-a') = (0-5)(2-5) = (7)(9) = 3$. Thus, $(b-a)(c-a) = (b'-a')(c'-a') = 3$. Since we have verified necessary conditions for $\{a,b,c\} = \{0,1,3\}$ and $\{a',b',c'\} = \{5,0,2\}$, the set classes containing $\{0,1,3\}$ and $\{5,0,3\}$ are dual.

It is interesting to note that the prevailing set class in the modern Western world, the set class 3-11, has a dual set class with which its voice-leading properties are algebraically compatible.

Theorem 5.2 (Voice-Leading Isomorphism on the Retention Group)**.** *Let γ and γ' be dual set classes, with corresponding groups $\mathcal{C}$ and $\mathcal{C}'$ generated by the retention operators. Then $\exists!\phi : \mathcal{C} \to \mathcal{C}'$, a nontrivial group isomorphism, such that*

$$\#_r \circ \phi = \#_r.$$

Before proving this main theorem, we begin with a definition and a lemma that will simplify its proof. Recall that if f is a coordinate system, then it has the property $T_n \circ f = f \circ T_n^*$.

Definition 5.3 (Riemannian Coordinate System)**.** *Let f be a coordinate system on a set class γ. f is a Riemannian coordinate system if*

$$I \circ f[(0,+)] = f \circ I^*[(0,+)].$$

In other words, if $f[(0,+)] = \{a,b,c\}$, then $f[(0,-)] = \{-a,-b,-c\}$.

Note that the coordinate system on the set class 3-11 that assigns to $(0,+)$ the C-Major triad and to $(0,-)$ the f-minor triad is a Riemannian coordinate system.

Lemma 5.4. *Let f be a Riemannian coordinate system on γ. Let $\{a,b,c\} \in \gamma$ such that $f^{-1}(\{a,b,c\}) = (0,+)$.*

(1) *Let $R_{b,c} \sim W_i$, $R_{a,c} \sim W_j$, and $R_{a,b} \sim W_k$. Then*

$$i = b + c$$

$$j = c + a$$

$$k = a + b$$

(2) *Also let* $R_a \sim W_l$, $R_b \sim W_m$, *and* $R_c \sim W_n$. *Then*

$$\begin{aligned} l &= 2a \\ m &= 2b \\ n &= 2c \end{aligned}$$

Likewise, if $f^{-1}(\{a,b,c\}) = (0,-)$, *then simply add a factor of* -1 *to one side of each equation given above.*

Proof. This proof is very similar to that of 4.3.

Part 1. We shall show that $k = b + a$ if $f^{-1}(\{a,b,c\}) = (0,+)$ and $k = -(b+a)$ if $f^{-1}(\{a,b,c\}) = (0,-)$. Then by symmetry, the other equations in (1) also follow by precisely the analogous arguments.

Note that $R_{a,b}(\{a,b,c\}) = \{a, b, a-c+b\}$ simply by the interval structure of γ. Also, $R_{a,b}(\{a,b,c\}) = f \circ W_k \circ f^{-1}(\{a,b,c\})$. Now, consider the case where $f^{-1}(\{a,b,c\}) = (0,+)$. Then,

$$\begin{aligned} R_{a,b}(\{a,b,c\}) &= f \circ W_k \circ f^{-1}(\{a,b,c\}) \\ &= f \circ W_k[(0,+)] \\ &= f[(k,-)] \\ &= f \circ T_k^*[(0,-)] \\ &= T_k \circ f[(0,-)] \\ &= T_k(\{-a,-b,-c\}) \\ &= \{-a+k, -b+k, -c+k\} \end{aligned}$$

because f is a Riemannian coordinate system. It now follows that

$$\{-a+k, -b+k, -c+k\} = \{a, b, a-c+b\}.$$

After considering every possible bijection between the two sets, the only system of equations that does not lead to a contradiction is the one that yields $k = b + a$. A similar computation for the case where $f^{-1}(\{a,b,c\}) = (0,-)$ yields $k = -(b+a)$.

Part 2. We shall show that $n = 2c$ if $f^{-1}(\{a,b,c\}) = (0,+)$ and $n = -2c$ if $f^{-1}(\{a,b,c\}) = (0,-)$. Then by symmetry, the other equations in (2) also follow by precisely the analogous arguments.

Note that $R_c(\{a,b,c\}) = \{c, 2c-b, 2c-a\}$ simply by the interval structure of γ. Also, $R_c(\{a,b,c\}) = f \circ W_n \circ f^{-1}(\{a,b,c\})$. Now, consider the case where $f^{-1}(\{a,b,c\}) = (0,+)$. Then,

$$\begin{aligned} R_c(\{a,b,c\}) &= f \circ W_n \circ f^{-1}(\{a,b,c\}) \\ &= f \circ W_n[(0,+)] \\ &= f[(n,-)] \\ &= f \circ T_n^*[(0,-)] \\ &= T_n \circ f[(0,-)] \\ &= T_n(\{-a,-b,-c\}) \\ &= \{-a+n, -b+n, -c+n\} \end{aligned}$$

because f is a Riemannian coordinate system. It now follows that

$$\{-a+n, -b+n, -c+n\} = \{c, 2c-b, 2c-a\}.$$

After considering every possible bijection between the two sets, the only system of equations that does not lead to a contradiction is the one that yields $n = 2c$. A similar computation for the case where $f^{-1}(\{a, b, c\}) = (0, -)$ yields $n = -2c$. $\square$

Now we can begin the proof of 5.2.

Proof of Theorem 5.2. This proof will proceed in stages, reducing each case to the next.

Step 1. First, we show that there exists a unique non-trivial isomorphism $\phi : \mathcal{C} \to \mathcal{C}'$ such that $\#_r = \#_r \circ \phi$ if and only if there exists a uniqe non-trivial automorphism $\Phi : \mathcal{R} \to \mathcal{R}$ such that $\#_r^* = \#_r^* \circ \Phi$ relative to arbitrary coordinate systems f and f' on γ and γ', respectively.

First, suppose $\exists! \phi : \mathcal{C} \to \mathcal{C}'$, an isomorphism, such that $\#_r = \#_r \circ \phi$. Then let $\Phi = \psi^{-1} \circ \phi \circ \psi$, where ψ is the isomorphism from 4.5. Then Φ is an isomorphism as it is a composition of isomorphisms. Thus, $\Phi : \mathcal{R} \to \mathcal{R}$ is an automorphism.

Now because $\#_r^* = \#_r \circ \psi$ by definition,

$$\begin{aligned}
\#_r^* \circ \Phi &= \#_r^* \circ \psi^{-1} \circ \phi \circ \psi \\
&= \#_r \circ \psi \circ \psi^{-1} \circ \phi \circ \psi \\
&= \#_r \circ \phi \circ \psi \\
&= \#_r \circ \psi \\
&= \#_r^*
\end{aligned}$$

For the converse existence, let $\phi = \psi \circ \Phi \circ \psi^{-1}$ where ψ is the isomorphism from 4.5. Then ϕ is an isomorphism as it is a composition of isomorphisms.

Now because $\#_r = \#_r^* \circ \psi^{-1}$,

$$\begin{aligned}
\#_r \circ \phi &= \#_r \circ \psi \circ \Phi \circ \psi^{-1} \\
&= \#_r^* \circ \psi^{-1} \circ \psi \circ \Phi \circ \psi^{-1} \\
&= \#_r^* \circ \Phi \circ \psi^{-1} \\
&= \#_r^* \circ \psi^{-1} \\
&= \#_r
\end{aligned}$$

Equivalence of uniqueness follows directly from the equivalence of existence.

Step 2. We now show that there exists a unique non-trivial automorphism $\Phi : \mathcal{R} \to \mathcal{R}$ such that $\#_r^* = \#_r^* \circ \Phi$ relative to any arbitrary coordinate systems f and f' on γ and γ', respectively, if and only if there exists a unique non-trivial automorphism $\Theta : \mathcal{R} \to \mathcal{R}$ such that $\#_r^* = \#_r^* \circ \Theta$ relative to two fixed Riemannian coordinate systems g and g' on γ and γ', respectively, with the property that $g^{-1}(\{a, b, c\}) = (0, +)$ and $g'^{-1}(\{a', b', c'\}) = (0, +)$.

First, suppose that $\exists! \Phi : \mathcal{R} \to \mathcal{R}$ such that $\#_r^* = \#_r^* \circ \Phi$ relative to any arbitrary coordinate systems f and f' on γ and γ'. Then clearly, this is also the case relative to the two fixed Riemannian coordinate systems g and g' on γ and γ' as g and g' are just a specific instance of an arbitrary coordinate system on γ and γ'.

For the converse, let Θ be the automorphism on $\mathcal{R}$ such that $\#_r^* = \#_r^* \circ \Phi$ relative to Riemannian coordinate systems g and g'. Now let f and f' be arbitrary coordinate systems on γ and γ', respectively. Then let Ω be the preservation automorphism from 3.1 relative to the ordered pair (f, g) and Ω' be the preservation automorphism relative to (g', f'). Let $\Phi = \Omega' \circ \Theta \circ \Omega$. Then Φ has the desired properties as you can check.

Step 3. Now it remains to show that $\exists! \Theta : \mathcal{R} \to \mathcal{R}$ such that $\#_r^* \circ \Theta = \#_r^*$ relative to Riemannian coordinate systems g and g' on γ and γ', respectively.

Now since γ and γ' are dual, let $\{a, b, c\} \in \gamma$ where $\gcd(b-a, 12) = 1$ and $\{a', b', c'\} \in \gamma'$ where $\gcd(b'-a', 12) = 1$.

Let g and g' be Riemannian coordinate systems on γ and γ'. Also suppose that $R_{b,c} \sim W_i$, $R_{a,c} \sim W_j$, $R_{a,b} \sim W_k$, $R_a \sim W_l$, $R_b \sim W_m$, and $R_c \sim W_n$ with respect to g. Likewise, suppose that $R_{b',c'} \sim W_{i'}$, $R_{a',c'} \sim W_{j'}$, $R_{a',b'} \sim W_{k'}$, $R_{a'} \sim W_{l'}$, $R_{b'} \sim W_{m'}$, and $R_{c'} \sim W_{n'}$ with respect to g'.

Now we work to construct Θ with the desired properties. Recall that we may completely specify Θ by its action on generators of $\mathcal{R}$. Because of 4.4, W_i and W_j generate $\mathcal{R}$. But because W_i and W_j are also double common-tone retention operators relative to g, they must map to another double common-tone retention operator relative to g'. So let $\Theta(W_i) = W_{\sigma(i)'}$ and $\Theta(W_j) = W_{\sigma(j)'}$, where σ is a permutation on the indices $\{i, j, k\}$. So for example, if σ is the identity permutation, then $\sigma(i)' = i'$. Note that this forces $\Theta(W_k)$ to equal $W_{\sigma(k)'}$ in order to preserve common-tone retention. Now we seek a permutation σ such that Θ is an isomorphism. Then, we must check that such a isomorphism also preserves operators with single common-tone retention.

Now, $W_k = (W_i W_j)^\alpha W_j$ for some integer α. Hence, $(i-j)\alpha + j = k$, which is equivalent to $\alpha(i-j) = (k-j)$. Now, because $\gcd(i-j, 12) = 1$ and all units in the ring $\mathbb{Z}_{12}$ have the property that $x^{-1} = x$, $\alpha = (k-j)(i-j)$.

Hence, if $\Theta(W_i) = W_{\sigma(i)'}$ and $\Theta(W_j) = W_{\sigma(j)'}$, then

$$\begin{aligned}\Theta(W_k) &= \Theta\big((W_i W_j)^{(k-j)(i-j)} W_j\big)\\ &= \big((W_{\sigma(i)'} W_{\sigma(j)'})^{(k-j)(i-j)} W_{\sigma(j)'}\big)\end{aligned}$$

But, we also have $\Theta(W_k) = W_{\sigma(k)'}$ in order to preserve double common-tone retention. Hence, the permutation σ must have the property that

$$(\sigma(i)' - \sigma(j)')(k-j)(i-j) + \sigma(j)' = \sigma(k)'. \tag{1}$$

So far, we have shown that the existence and uniqueness of an automorphism Θ that preserves double common-tone retention is equivalent to the existence and uniqueness of σ such that the above equation holds and that Θ is a bijection. We will check single common-tone retention later.

First, we show existence of such a σ. Let σ be the identity permutation. Then the equation reduces to $(i'-j')(k-j)(i-j) + j' = k'$. We shall show that the following equation holds:

$$(i'-j')(k-j)(i-j) = (k'-j').$$

Now since $g^{-1}(\{a, b, c\}) = (0, +)$ and $g'^{-1}(\{a', b', c'\}) = (0, +)$, lemma 4.3 implies that the above equation can be rewritten as

$$(b'-a')(b-c)(b-a) = (b'-c').$$

Once again, recalling that all units in the ring $\mathbb{Z}_{12}$ have the property that $x^{-1} = x$, we can rewrite the equation as

$$(b-a)(b-c) = (b'-a')(b'-c').$$

which is true by duality, proving existence.

For uniqueness, one may pursue analogous arguments as above and show that the other permutations on the letters $\{i, j, k\}$ do not work for equation (1).

Finally, we check that Θ is an automorphism. Since Θ sends the generators W_i and W_j to the generators W_i' and W_j', it is an isomorphism.

Step 4. We have shown that duality implies the uniqueness and existence of an automorphism that preserves double common-tone retention operators. Now we show that it also preserves single common-tone retention operators. We shall show that $\Theta(W_l) = W_{l'}$, $\Theta(W_m) = W_{m'}$, and $\Theta(W_n) = W_{n'}$.

Note that $W_l = (W_iW_j)^\alpha W_j$, where $\alpha = (l-j)(i-j)$. Then

$$\begin{aligned}\Theta(W_l) &= \Theta\big((W_iW_j)^{(l-j)(i-j)}W_j\big)\\ &= \big((W_{i'}W_{j'})^{(l-j)(i-j)}W_{j'}\big).\end{aligned}$$

Hence, we must now show that

$$(i'-j')(l-j)(i-j) + j' = l'$$

which is equivalent to

$$(i'-j')(l-j)(i-j) = l'-j'.$$

Recalling once more that the units in the ring $\mathbb{Z}_{12}$ have the property that $x^{-1} = x$,

$$(i'-j')(l'-j') = (i-j)(l-j). \tag{2}$$

Now, because $g^{-1}(\{a, b, c\}) = (0, +)$ and $g'^{-1}(\{a', b', c'\}) = (0, +)$, lemma 4.3 implies that this is equivalent to showing

$$(b'-a')(l'-j') = (b-a)(l-j).$$

But because g and g' are Riemannian coordinate systems, it is also true by lemma 5.4, $l = 2a$ and $j = c + a$. Likewise, $l' = 2a'$ and $j' = c' + a'$. Thus, $(l-j) = (2a-(c+a)) = (a-c)$ and $(l'-j') = (2a'-(c'+a')) = (a'-c')$. Now it follows that we must show

$$(b'-a')(a'-c') = (b-a)(a-c) \tag{3}$$

which follows once again by duality.

Showing $\Theta(W_n) = W_{n'}$ is exactly analogous. However one cannot show $\Theta(W_m) = W_{m'}$ in precisely the same manner as the final equation yielded is

$$(b'-a')(2b'-c'+a') = (b-a)(2b-c+a)$$

which does not necessarily follow from duality.

Hence to show $\Theta(W_m) = W_{m'}$, we use the fact that $l'-m' = 2(a'-b')$ by lemma 4.3. Indeed, showing $\Theta(W_m) = W_{m'}$ is equivalent to showing that $(i'-j')(m-j)(i-j)+j' = m'$. But we already know that $(i'-j')(l-j)(i-j)+j' = l'$ because $\Theta(W_l) = W_{l'}$ and so it suffices to show that $l'-m' = (i'-j')(l-j)(i-j)+j' - [(i'-j')(m-j)(i-j)+j'] = 2(a'-b')$. By lemma 4.3, this is equivalent to showing

$$(b'-a')(l-j)(b-a) - (b'-a')(m-j)(b-a) = 2(a'-b').$$

Now,

$$
\begin{aligned}
(b'-a')(l-j)(b-a)-(b'-a')(m-j)(b-a) &= (b'-a')(b-a)[(l-j)-(m-j)] \\
&= (b'-a')(b-a)(l-m) \\
&= (b'-a')(b-a)(2(a-b)) \\
&= (a'-b')(a-b)(2(a-b)) \\
&= (a'-b')2(a-b)^2 \\
&= 2(a'-b')
\end{aligned}
$$

□

Corollary 5.5 (Voice-Leading Automorphism on the Riemannian Group)**.** *Let γ and γ' be dual set classes with coordinate systems f and f', respectively. Then $\exists \Phi : \mathcal{R} \to \mathcal{R}$, an automorphism, such that*

$$\#_r^* \circ \Phi = \#_r^*.$$

6. Extension to Tetrachords

One way to generalize is to call any two set classes *dual* if there exists an automorphism between them that preserves common-tone retention.

Definition 6.1 (Dual set classes)**.** *Consider any two set classes γ and γ' with corresponding groups $\mathcal{C}$ and $\mathcal{C}'$. Then we say that γ and γ' are dual if there exists a nontrivial group isomorphism $\phi : \mathcal{C} \to \mathcal{C}'$, such that*

$$\#_r \circ \phi = \#_r.$$

Now we proceed to compute and classify all dual set class types for tetrachords. We shall see that, in general, the voice-leading isomorphism is unique (even though we only set out initially to find at least one), except in a single exception where there are two voice leading isomorphisms.

The first step is to enumerate all common-tone retention operators. This is a matter of trivial computation (if even that), thus we simply provide a table listing all of them. As with trichords, we choose to work only with set classes with 24 elements. For convenience, we shall display the retention operators as Riemannian operators by fixing a Riemannian coordinate system f on the set classes, where $f(0, +)$ is the prime form of the set class. We list them in the order of increasing number of triple common-tone retention operators, then in the order of increasing number of double common-tone retention operators.

set class (Prime Form)	Triple Common-tone Retention	Double Common-tone Retention
{0,1,4,6}	None	$S_6, W_0, W_1, W_4, W_5, W_6, W_7, W_{10}$
{0,1,3,7}	None	$S_6, W_1, W_2, W_3, W_4, W_7, W_8, W_{10}$
{0,1,2,5}	W_2	$S_1, S_{11}, W_1, W_3, W_5, W_6, W_7$
{0,1,3,5}	W_6	$S_2, S_{10}, W_1, W_3, W_4, W_5, W_8$
{0,2,3,7}	W_2	$S_5, S_7, W_3, W_5, W_7, W_9, W_{10}$
{0,1,2,6}	W_2	$S_1, S_6, S_{11}, W_0, W_1, W_3, W_6, W_7, W_8$
{0,2,3,6}	W_6	$S_3, S_6, S_9, W_0, W_2, W_3, W_5, W_8, W_9$
{0,1,3,6}	W_6	$S_3, S_6, S_9, W_0, W_1, W_3, W_4, W_7, W_9$
{0,1,5,7}	W_0	$S_5, S_6, S_7, W_1, W_2, W_5, W_6, W_7, W_8$
{0,1,4,7}	W_8	$S_3, S_6, S_9, W_1, W_2, W_4, W_5, W_7, W_{11}$
{0,2,5,8}	W_2	$S_3, S_6, S_9, W_4, W_5, W_7, W_8, W_{10}, W_{11}$
{0,1,2,4}	W_2, W_4	$S_1, S_2, S_{10}, S_{11}, W_1, W_3, W_5, W_6$
{0,2,4,7}	W_2, W_4	$S_2, S_5, S_7, S_{10}, W_6, W_7, W_9, W_{11}$
{0,1,4,8}	S_4, W_0W_4, W_8	S_8, W_1, W_5, W_9

Now, we provide the list of dual tetrachordal set classes.

Theorem 6.2 (Tetrachordal Dual Set Classes)**.**

set class	*Dual set class*	*VL Automorphism's Action on Generators*
{0,1,4,6}	*{0,1,3,7}*	$W_0 \mapsto W_2$ *and* $W_1 \mapsto W_7$
{0,1,2,5}	*{0,2,3,7}*	$W_2 \mapsto W_2$ *and* $W_3 \mapsto W_7$
{0,1,3,5}	*{0,1,3,5}*	$W_6 \mapsto W_6$ *and* $W_5 \mapsto W_1$
{0,1,2,6}	*{0,1,5,7}*	$W_2 \mapsto W_0$ *and* $W_3 \mapsto W_5$
{0,2,3,6}	*{0,2,5,8}*	$W_6 \mapsto W_2$ *and* $W_5 \mapsto W_7$
{0,1,3,6}	*{0,1,3,6}*	$W_6 \mapsto W_6$ *and* $W_7 \mapsto W_1$
{0,1,4,7}	*{0,1,4,7}*	$W_8 \mapsto W_8$ *and* $W_7 \mapsto W_1$
{0,1,2,4}	*{0,2,4,7}*	$W_2 \mapsto W_2$ *and* $W_1 \mapsto W_7$
{0,1,4,8}	*{0,1,4,8}*	

Recall that dual set classes have isomorphic voice-leading structures, in that there is a non-trivial group isomorphism that preserves common-tone retention. We note that the voice-leading isomorphism is always unique, except in the one exception of the dual set classes $\{0, 1, 4, 6\}$ and $\{0, 1, 3, 7\}$. Note that these are also precisely the set classes that don't have triple-common tone retention operators.

Proof. As a general guideline, note that since the voice-leading isomorphism is a bijection, dual set classes *must* have the same number of both triple and double common-tone retention operators. This limits the cases we have to check dramatically.

Note that there are only two set classes with no triple common-tone operators: $\{0, 1, 4, 6\}$ and $\{0, 1, 3, 7\}$. In fact, these set classes are dual. To see this, we need to find an isomorphism Φ that maps all of the double common-tone retention operators on $\{0, 1, 4, 6\}$ to that on $\{0, 1, 3, 7\}$. While this instance is easy enough for trial and error, we establish a formal mechanism for use later on. We need an isomorphism that maps everything in $D_{\{0,1,4,6\}} = \{W_0, W_1, W_4, W_5, W_6, W_7, W_{10}\}$ to everything in $D_{\{0,1,3,7\}} = \{W_1, W_2, W_3, W_4, W_7, W_8, W_{10}\}$.

Now, $W_1, W_0 \in D_{\{0,1,4,6\}}$ generate all of $\mathcal{R}$. Thus, fixing where the isomorphism Φ sends these two generators fixes the entire isomorphism. So, in general, let

$W_0 \mapsto W_x$ and $W_1 \mapsto W_y$. Then, since $W_n = (W_1W_0)^n W_0$,

$$\begin{aligned}\Phi(W_n) &= \Phi((W_1W_0)^n W_0)\\ &= (W_yW_x)^n W_x\\ &= (S_{y-x})^n W_x\\ &= W_{n(y-x)+x}.\end{aligned}$$

Thus, $W_n \mapsto W_{n(y-x)+x}$. We take this as an equation on just the indices: $n \mapsto n(y-x)+x$. Then, we need the indices in $D_{\{0,1,4,6\}}$ to map one-to-one and onto the indices in $D_{\{0,1,3,7\}}$.

To summarize, we need to pick $W_x, W_y \in D_{\{0,1,3,7\}}$ such that $n \mapsto n(y-x)+x$ is a bijection on the indices from $D_{\{0,1,4,6\}}$ to $D_{\{0,1,3,7\}}$.

At this point, we use a Mathematica code to go through every single mapping between these two sets possible and tell us which choices of x and y make the index equation a bijection, so that we have a valid isomorphism that preserves common-tone retention. The Mathematica code and its results are duplicated below:

```
In[135]:= autocheck[G_, H_] := Module[{A, P, i, x, y, T, m},
            A = {};
            P = Subsets[H, {2}];
            m = Length[P];
            i = 1;
            While[i ≤ m,
              {x, y} = P[[i]]; T = Mod[G (y - x) + x, 12];
             A = Append[A, {{x, y}, Sort[T] == Sort[H]}];
             {y, x} = P[[i]]; T = Mod[G (y - x) + x, 12];
             A = Append[A, {{x, y}, Sort[T] == Sort[H]}]; i++];
            A]

In[136]:= autocheck[{0, 1, 4, 5, 6, 7, 10}, {1, 2, 3, 4, 7, 8, 10}]

Out[136]= {{{1, 2}, False}, {{2, 1}, False}, {{1, 3}, False},
           {{3, 1}, False}, {{1, 4}, False}, {{4, 1}, False}, {{1, 7}, False},
           {{7, 1}, False}, {{1, 8}, False}, {{8, 1}, False}, {{1, 10}, False},
           {{10, 1}, False}, {{2, 3}, False}, {{3, 2}, False}, {{2, 4}, False},
           {{4, 2}, False}, {{2, 7}, True}, {{7, 2}, False}, {{2, 8}, False},
           {{8, 2}, False}, {{2, 10}, False}, {{10, 2}, False}, {{3, 4}, False},
           {{4, 3}, False}, {{3, 7}, False}, {{7, 3}, False}, {{3, 8}, False},
           {{8, 3}, False}, {{3, 10}, False}, {{10, 3}, False}, {{4, 7}, False},
           {{7, 4}, False}, {{4, 8}, False}, {{8, 4}, False}, {{4, 10}, False},
           {{10, 4}, False}, {{7, 8}, False}, {{8, 7}, True}, {{7, 10}, False},
           {{10, 7}, False}, {{8, 10}, False}, {{10, 8}, False}}
```

Figure 1 (Mathematica Code)

In the output, notice that there are precisely two ordered pairs for which Mathematica declares "True": $\{2,7\}$ and $\{8,7\}$. In other words, the assignments $x \mapsto 2$ a and $y \mapsto 7$ yields one valid isomorphism and the assignments $x \mapsto 8$ and $y \mapsto 7$ is another valid isomorphism. Thus, there are two voice-leading isomorphisms on these dual set classes. We note that this is the *only* case where the voice-leading isomorphism is not unique, and one reason might be that these set classes have no triple common-tone retention operators, so more degrees of freedom for a isomorphism that preserves common-tone retention.

Now we move to a more rich class of dual set classes: those with precisely one triple common-tone retention. Here, we employ a similar strategy. Here, since there is only one triple common-tone retention operator, and since the voice-leading isomorphism must take a triple common-tone retention operator to a triple-common tone retention operator, the action of our isomorphism is already fixed for one element.

In general, let W_a be the triple common-tone retention operator in one set class, and W_b be the triple common-tone retention operator in another set class, so that $W_a \mapsto W_b$. Now we need to determine the action Φ on a second element so that its action is completely determined. We must pick this second element so that W_a and this second operator generate all of $\mathcal{R}$, otherwise the isomorphism is not completely determined. One way to guarantee this is to pick the second element to be either W_{a+1} or W_{a-1}. (Think of this as two reflections that are a single axis off from one another in the dihedral group.) Fortunately, for all set classes that we will be considering in this case, if W_a is a triple common-tone operator, then either W_{a+1} or W_{a-1} is a double common-tone retention operator. So, in general, either let $W_{a-1} \mapsto W_x$ or $W_{a+1} \mapsto W_x$. By similar analysis as above, this means that either $W_n \mapsto W_{(x-b)(n-a)+b}$ or $W_{(x-b)(a-n)+b}$, respectively.

Now we run a very similar Mathematica code to see which bijections between indices of double common-tone retention operators obey the index equation given above. Note that there are two different codes given below: one titled *autocheckp* and another titled *autocheckm*. The two separate codes are necessary, as some set classes have retention operators corresponding to W_a and W_{a-1}, whereas others have W_a and W_{a+1}.

```
In[84]:= autocheckp[G_, H_, a_, b_] := Module[{A, P, i, x, T, m},
           A = {};
           m = Length[H];
           i = 1;
           While[i ≤ m,
            x = H[[i]]; T = Mod[(x - b) (G - a) + b, 12];
            A = Append[A, {{x}, Sort[T] == Sort[H]}];
            i ++];
           A]
         autocheckm[G_, H_, a_, b_] := Module[{A, P, i, x, T, m},
           A = {};
           m = Length[H];
           i = 1;
           While[i ≤ m,
            x = H[[i]]; T = Mod[(x - b) (a - G) + b, 12];
            A = Append[A, {{x}, Sort[T] == Sort[H]}];
            i ++];
           A]

In[74]:= A1 = {1, 3, 5, 6, 7};
         A2 = {0, 1, 3, 6, 7, 8};
         A3 = {1, 3, 4, 5, 8};
         A4 = {0, 2, 3, 5, 8, 9};
         A5 = {0, 1, 3, 4, 7, 9};
         A6 = {3, 5, 7, 9, 10};
         A7 = {1, 2, 5, 6, 7, 8};
         A8 = {1, 2, 4, 5, 7, 11};
         A9 = {4, 5, 7, 8, 10, 11};
```

Figure 2 (Mathematica Code)

```
In[89]:= autocheckp[A1, A1, 2, 2]
         autocheckp[A1, A3, 2, 6]
         autocheckp[A1, A6, 2, 2]

Out[89]= {{{1}, False}, {{3}, True}, {{5}, False}, {{6}, False}, {{7}, False}}

Out[90]= {{{1}, False}, {{3}, False}, {{4}, False}, {{5}, False}, {{8}, False}}

Out[91]= {{{3}, False}, {{5}, False}, {{7}, True}, {{9}, False}, {{10}, False}}

In[92]:= autocheckm[A3, A1, 6, 2]
         autocheckm[A3, A3, 6, 6]
         autocheckm[A3, A6, 6, 2]

Out[92]= {{{1}, False}, {{3}, False}, {{5}, False}, {{6}, False}, {{7}, False}}

Out[93]= {{{1}, True}, {{3}, False}, {{4}, False}, {{5}, True}, {{8}, False}}

Out[94]= {{{3}, False}, {{5}, False}, {{7}, False}, {{9}, False}, {{10}, False}}

In[95]:= autocheckp[A6, A1, 2, 2]
         autocheckp[A6, A3, 2, 6]
         autocheckp[A6, A6, 2, 2]

Out[95]= {{{1}, False}, {{3}, False}, {{5}, False}, {{6}, False}, {{7}, True}}

Out[96]= {{{1}, False}, {{3}, False}, {{4}, False}, {{5}, False}, {{8}, False}}

Out[97]= {{{3}, True}, {{5}, False}, {{7}, False}, {{9}, False}, {{10}, False}}
```

Figure 3 (Mathematica Code)

```
In[104]:= autocheckm[A4, A2, 6, 2]
          autocheckm[A4, A4, 6, 6]
          autocheckm[A4, A5, 6, 6]
          autocheckm[A4, A7, 6, 0]
          autocheckm[A4, A8, 6, 8]
          autocheckm[A4, A9, 6, 2]

Out[104]= {{{0}, False}, {{1}, False}, {{3}, False},
           {{6}, False}, {{7}, False}, {{8}, False}}

Out[105]= {{{0}, False}, {{2}, False}, {{3}, False},
           {{5}, True}, {{8}, False}, {{9}, False}}

Out[106]= {{{0}, False}, {{1}, False}, {{3}, False},
           {{4}, False}, {{7}, False}, {{9}, False}}

Out[107]= {{{1}, False}, {{2}, False}, {{5}, False},
           {{6}, False}, {{7}, False}, {{8}, False}}

Out[108]= {{{1}, False}, {{2}, False}, {{4}, False},
           {{5}, False}, {{7}, False}, {{11}, False}}

Out[109]= {{{4}, False}, {{5}, False}, {{7}, True},
           {{8}, False}, {{10}, False}, {{11}, False}}

In[110]:= autocheckp[A5, A2, 6, 2]
          autocheckp[A5, A4, 6, 6]
          autocheckp[A5, A5, 6, 6]
          autocheckp[A5, A7, 6, 0]
          autocheckp[A5, A8, 6, 8]
          autocheckp[A5, A9, 6, 2]

Out[110]= {{{0}, False}, {{1}, False}, {{3}, False},
           {{6}, False}, {{7}, False}, {{8}, False}}

Out[111]= {{{0}, False}, {{2}, False}, {{3}, False},
           {{5}, False}, {{8}, False}, {{9}, False}}

Out[112]= {{{0}, False}, {{1}, True}, {{3}, False},
           {{4}, False}, {{7}, True}, {{9}, False}}

Out[113]= {{{1}, False}, {{2}, False}, {{5}, False},
           {{6}, False}, {{7}, False}, {{8}, False}}

Out[114]= {{{1}, False}, {{2}, False}, {{4}, False},
           {{5}, False}, {{7}, False}, {{11}, False}}

Out[115]= {{{4}, False}, {{5}, False}, {{7}, False},
           {{8}, False}, {{10}, False}, {{11}, False}}
```

Figure 4 (Mathematica Code)

```
In[116]:= autocheckp[A7, A2, 0, 2]
          autocheckp[A7, A4, 0, 6]
          autocheckp[A7, A5, 0, 6]
          autocheckp[A7, A7, 0, 0]
          autocheckp[A7, A8, 0, 8]
          autocheckp[A7, A9, 0, 2]

Out[116]= {{{0}, False}, {{1}, False}, {{3}, False},
           {{6}, False}, {{7}, True}, {{8}, False}}

Out[117]= {{{0}, False}, {{2}, False}, {{3}, False},
           {{5}, False}, {{8}, False}, {{9}, False}}

Out[118]= {{{0}, False}, {{1}, False}, {{3}, False},
           {{4}, False}, {{7}, False}, {{9}, False}}

Out[119]= {{{1}, True}, {{2}, False}, {{5}, False},
           {{6}, False}, {{7}, False}, {{8}, False}}

Out[120]= {{{1}, False}, {{2}, False}, {{4}, False},
           {{5}, False}, {{7}, False}, {{11}, False}}

Out[121]= {{{4}, False}, {{5}, False}, {{7}, False},
           {{8}, False}, {{10}, False}, {{11}, False}}

In[122]:= autocheckm[A8, A2, 8, 2]
          autocheckm[A8, A4, 8, 6]
          autocheckm[A8, A5, 8, 6]
          autocheckm[A8, A7, 8, 0]
          autocheckm[A8, A8, 8, 8]
          autocheckm[A8, A9, 8, 2]

Out[122]= {{{0}, False}, {{1}, False}, {{3}, False},
           {{6}, False}, {{7}, False}, {{8}, False}}

Out[123]= {{{0}, False}, {{2}, False}, {{3}, False},
           {{5}, False}, {{8}, False}, {{9}, False}}

Out[124]= {{{0}, False}, {{1}, False}, {{3}, False},
           {{4}, False}, {{7}, False}, {{9}, False}}

Out[125]= {{{1}, False}, {{2}, False}, {{5}, False},
           {{6}, False}, {{7}, False}, {{8}, False}}

Out[126]= {{{1}, True}, {{2}, False}, {{4}, False},
           {{5}, False}, {{7}, True}, {{11}, False}}

Out[127]= {{{4}, False}, {{5}, False}, {{7}, False},
           {{8}, False}, {{10}, False}, {{11}, False}}
```

Figure 5 (Mathematica Code)

If you do not want to bother figuring out how the Mathematica code works, here is an easy way to decipher its output: for example, the input *autocheckm*$[A8, A2, 8, 2]$ checks if there is a voice-leading isomorphism between set classes labeled A8 and A2 (these are given further up in the code). (The 8 and 2 mean that W_8 and W_2 are the triple common-tone retention operators in the set classes A8 and A2, respectively). Now, in the result, if you see a "True", that means we have found an isomorphism, and the number to the left of it is where x gets sent to in the isomorphism.

The above computations demonstrate a remarkable and satisfying phenomenon that is the theme of this paper: in each case, result, there are precisely two "True" statements: one corresponding to the trivial identity isomorphism, and one corresponding to our voice-leading isomorphism. In other words, we can plainly observe that the voice-leading isomorphism is unique.

Now we move to the case where we have 2 triple common-tone retention operators. For this case, we are back to only having to check 2 set classes. Here, the analysis is only slightly different because we have two triple common-tone retention operators. Nevertheless, the methods are precisely analogous. In the Mathematica code given below, notice that we have 2 different codes: *autocheck1* and *autocheck2*. This is because we have 2 possibilities for where to send the triple common-tone retention operator W_2. Either $W_2 \mapsto W_2$ or $W_2 \mapsto W_4$. In each case, we then let $W_1 \mapsto W_x$. Since W_1 is a triple common-tone retention operator in $\{0, 1, 2, 4\}$, so must W_x in $\{0, 2, 4, 7\}$.

```
In[130]:= autocheck1[G_, H_] := Module[{A, P, i, x, T, m},
            A = {};
            m = Length[H];
            i = 1;
            While[i ≤ m,
              x = H[[i]]; T = Mod[(2 - x) (G - 1) + x, 12];
             A = Append[A, {{x}, Sort[T] == Sort[H]}];
             i++];
            A]

In[131]:= autocheck2[G_, H_] := Module[{A, P, i, x, T, m},
            A = {};
            m = Length[H];
            i = 1;
            While[i ≤ m,
              x = H[[i]]; T = Mod[(4 - x) (G - 1) + x, 12];
             A = Append[A, {{x}, Sort[T] == Sort[H]}];
             i++];
            A]

In[133]:= autocheck1[{1, 3, 5, 6}, {6, 7, 9, 11}]

Out[133]= {{{6}, False}, {{7}, True}, {{9}, False}, {{11}, False}}

In[134]:= autocheck2[{1, 3, 5, 6}, {6, 7, 9, 11}]

Out[134]= {{{6}, False}, {{7}, False}, {{9}, False}, {{11}, False}}
```

Figure 6 (Mathematica Code)

Once again, observe that there is only one valid isomorphism: the one that maps $x \mapsto 7$.

□

Thus, we have succeeded in classifying all tetrachordal set classes with 24 elements in terms of their duality. Just like for trichords, dual set classes have one and only one voice-leading automorphism (with one exception where there are no triple common-tone retention operators).

However, there is one technical caveat. Upon a careful inspection of the above analysis, it is evident that we always assumed that a W_n would get mapped to some other W_m. Nevertheless, we still discovered all voice-leading automorphisms. There is no a priori reason to believe that an automorphism that maps W_n to an S_m would not be a voice-leading automorphism. However, an inspection of the data suggests that this never occurs.

7. Thoughts on Generalizations

Because the voice-leading automorphism continued to be uniquely determined for tetrachords, there is strong evidence to suggest that the likewise will hold in arbitrary set classes. First, these will have to be computed, then a proof that shows why such isomorphisms exist would be in order. I have already provided a proof for the trichordal case, but it is not clear that the proof will easily generalize.

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