# STOCHASTIC INTEREST RATE DISTRIBUTIONS VIA GREEN'S FUNCTIONS

MAXX HYEOK CHO, SWARTHMORE COLLEGE
SCOTT GLASGOW, BRIGHAM YOUNG UNIVERSITY

Abstract. The Term Structure Interest Rate Model of Cox, Ingersoll, and Ross (CIR) is a commonly used model when considering interest rates as a random variable. However, the CIR stochastic differential equation has no known closed-form solutions. In this paper, a Green's function approach to solving the differential equation not commonly employed outside of mathematical physics is used to derive a closed-form formula for the density function of the interest rate process that agrees with the distribution Cox, Ingersoll, and Ross originally discovered. Our formulation of this distribution is entirely rigorous and logically sound, providing an ansatz-free derivation.

## 1. The Cox-Ingersoll-Ross Model for Interest Rates

The Cox-Ingersoll-Ross (CIR) model for interest rates is one of many well-known stochastic differential equations that model the evolution of interest rates. The model was originally published by Cox, Ingersoll, and Ross in the 1985 paper titled *A Theory of the Term Structure of Interest Rates*. This affine stochastic differential equation does not have any closed-form solutions. However, the related partial differential equation that results from the Feynman-Kac Theorem has also been well-studied, including its explicit closed-form solution. In this first section, we discuss some well-known features of the CIR stochastic differential equation, including its related partial differential equation.

### 1.1. The CIR Stochastic Differential Equation.

The CIR stochastic differential equation models the evolution of interest rates. Notably, the equation does not permit negative interest rate values.

**Equation 1.1** (The CIR Stochastic Differential Equation)**.**

$$dX_\cdot(t) = \big(a + bX_\cdot(t)\big)dt + c\sqrt{X_\cdot(t)}d\tilde{W}_\cdot(t)$$

*The constants $a$ and $b$ are positive and the constant $b$ is negative. Some authors choose to change the equation itself so that all three constants are positive.*

The variable $X_\cdot(t)$ is the interest rate at time $t$. The subscript dot on the variable indicates that, in addition to the time dependence, the function is also defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We indicate the random nature of the variable only by a subtle dot because, in our analysis, it is convenient to think of the $\Omega$ dependence as fixed. The function $\tilde{W}_\cdot(t)$ is Brownian Motion (or a Weiner Process)

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under the risk-neutral measure. Note that if this volatile term is set to zero, the resulting deterministic differential equation will have an equilibrium value of $-\frac{a}{b}$. In other words, the solution to the deterministic part will approach the equilibrium value of $-\frac{a}{b}$ as time approaches infinity. Considered together with the volatile term, we can heuristically see that the interest rate evolving according to equation 1.1 will "oscillate" towards the value $-\frac{a}{b}$ with the volatile term shifting the interest rate about this equilibrium. Also note that this equilibrium value is positive, as it must be.

1.2. **The CIR Partial Differential Equation.** Now we seek to find a partial differential equation that is related to the CIR stochastic differential equation. There are many reasons to seek such an equation. One reason might be that deterministic equations are much easier to work with. But, as we shall see later on, finding the partial differential equation also helps us derive the distribution of the random variable. The following well-known theorem produces the desired deterministic equation:

**Theorem 1.2** (The Feynman-Kac Theorem)**.** *Consider the stochastic differential equation*

$$dX.(u) = \beta\big(u, X.(u)\big) + \gamma\big(u, X.(u)\big)dW.(u).$$

*Let $f(x)$ be a borel-measurable function. Fix some $t > 0$ and let $s \in [0, t]$ be given. Also let $g(s, x) = \mathbb{E}\{f\big(X.(t)\big)|X.(s) = x\}$. Then $g(s, x)$ satisfies the deterministic partial differential equation:*

$$g_s(s, x) + \beta(s, x)g_x(s, x) + \frac{1}{2}\gamma^2(s, x)g_{xx}(s, x) = 0$$

*with the terminal condition*

$$g(t, x) = f(x)$$

*for all $x$.*

No proof of the Feynman-Kac theorem will be given. It is sufficient to realize that the stochastic differential equation has been rendered into a deterministic one by constructing a new function, $g(s, x)$, which turns out to be martingale when it takes random values for $x$. This new function, which is the expectation of the random variable evolving according to the stochastic differential equation, will prove useful later.

When one considers the CIR stochastic differential equation specifically, it is easy to to see that the Feynman-Kac theorem yields the following deterministic partial differential equation:

**Equation 1.3** (CIR Partial Differential Equation)**.**

$$g_s(s, x) + (a + bx)g_x(s, x) + \frac{c^2}{2}xg_{xx}(s, x) = 0, \quad (s, x) \in [0, t] \times [0, \infty)$$

$$g(t, x) = f(x)$$

*where $g(s, x) = \mathbb{E}\{f\big(X.(t)\big)|X.(s) = x\}$.*

## 2. Green's Function Approach to the Distribution

2.1. **Deriving the Density Function by Differential Operators.** In mathematical physics, the Green's function approach to solving partial differential equations is commonly used in Quantum Mechanics. We apply this technique to the CIR Partial Differential Equation (Equation 1.3) and discover that it naturally leads to a formulation of a function that may be interpreted as the distribution of the random variable evolving according to the CIR Stochastic Differential Equation (Equation 1.1).

First, we define a differential operator that naturally arises in the Green's function analysis of the CIR partial differential equation (Equation 1.3). This operator is defined in advance for clarity.

**Definition 2.1.**

$$H := (a + bx)\partial_x + \frac{c^2}{2}x(\partial_x)^2$$

*This is a second-order differential operator.*

Now we seek to characterize the solutions of the CIR partial differential equation (Equation 1.3) in a way that will help us derive the density function of the interest rate evolving according to the CIR stochastic differential equation (Equation 1.1). The characterization we need expresses the partial differential equation's solution, $g(s, x)$, as an operator acing on the initial condition, $f(x)$.

**Theorem 2.2** (Solution to the CIR partial differential equation)**.** *Suppose a function $g(s, x)$ satisfies the CIR partial differential equation (Equation 1.3). In other words, it is a solution of the partial differential equation*

$$g_s(s, x) + (a + bx)g_x(s, x) + \frac{c^2}{2}xg_{xx}(s, x) = 0, \quad (s, x) \in [0, t] \times [0, \infty)$$
$$g(t, x) = f(x).$$

*Then, the function $g(s, x)$ can be written as*

$$g(s, x) = \exp\left[(t - s)H\right] \cdot f(x).$$

*Proof.* Consider a Taylor expansion of $g(s, x)$ in the $s$ direction about $s = t$, holding $x$ fixed:

(1)
$$g(s,x) = g(t,x) + g_s(t,x)(s-t) + \frac{1}{2}g_{ss}(t,x)(s-t)^2 + \cdots + \frac{1}{n!}g_{\underbrace{s\cdots s}_{n}}(t,x)(s-t)^n + \cdots.$$

Notice that the power series contains derivatives of $g(s, x)$ with respect to $s$, which are related to derivatives of $g(s, x)$ with respect to $x$ in a specific way given by the partial differential equation under consideration. The goal is to change equation 1 so that it only contains derivatives with respect to $x$.

Solving the differential equation (Equation 1.3) for $g_s(s, x)$,

$$\partial_s g = -(a + bx)\partial_x g - \frac{c^2}{2}x\partial_x\partial_x g. \tag{2}$$

where we have changed to the operator notation of derivatives, and suppressed the arguments of $g$. If we take another derivative with respect to $s$ on both sides of equation 2, we obtain

$$\partial_s\partial_s g = -(a + bx)\partial_x\partial_s g - \frac{c^2}{2}x\partial_x\partial_x\partial_s g. \tag{3}$$

But note that this new equation has terms of $\partial_s g$, for which we have already derived a formula (Equation 2). So plugging this back in yields

$$\begin{aligned}\partial_s\partial_s g &= -(a+bx)\partial_x\big(-(a+bx)\partial_x g - \frac{c^2}{2}x\partial_x\partial_x g\big) - \frac{c^2}{2}x\partial_x\partial_x\big(-(a+bx)\partial_x g - \frac{c^2}{2}x\partial_x\partial_x g\big)\\ &= \big(-(a+bx)\partial_x - \frac{c^2}{2}x\partial_x\partial_x\big)\big(-(a+bx)\partial_x g - \frac{c^2}{2}x\partial_x\partial_x g\big)\\ &= \big(-(a+bx)\partial_x g - \frac{c^2}{2}x\partial_x\partial_x g\big)^2\\ &= (-1)^2\big((a+bx)\partial_x g + \frac{c^2}{2}x\partial_x\partial_x g\big)^2\\ &= \big((a+bx)\partial_x g + \frac{c^2}{2}x\partial_x\partial_x g\big)^2. \end{aligned} \tag{4}$$

With repeated differentiation on both sides with respect to $s$, one can easily verify that

$$(\partial_s)^n g = (-1)^n\big((a+bx)\partial_x g + \frac{c^2}{2}x\partial_x\partial_x g\big)^n. \tag{5}$$

This result can be used to replace all of the derivatives in $s$ to derivatives in $x$ in the power series representation of $g(s,x)$:

$$\begin{aligned} g(s,x) &= g(t,x) + (-1)\big((a+bx)\partial_x g(t,x) + \frac{c^2}{2}x\partial_x\partial_x g(t,x)\big)(s-t) +\\ &\quad \frac{1}{2}(-1)^2\big(-(a+bx)\partial_x g(t,x) - \frac{c^2}{2}x\partial_x\partial_x g(t,x)\big)^2(s-t)^2 +\\ &\quad \cdots + \frac{1}{n!}(-1)^n\big((a+bx)\partial_x g(t,x) + \frac{c^2}{2}x\partial_x\partial_x g(t,x)\big)^n(s-t)^n + \cdots\\ &= g(t,x) + \big((a+bx)\partial_x g(t,x) + \frac{c^2}{2}x\partial_x\partial_x g(t,x)\big)(t-s) +\\ &\quad \frac{1}{2}\big(-(a+bx)\partial_x g(t,x) - \frac{c^2}{2}x\partial_x\partial_x g(t,x)\big)^2(t-s)^2 +\\ &\quad \cdots + \frac{1}{n!}\big((a+bx)\partial_x g(t,x) + \frac{c^2}{2}x\partial_x\partial_x g(t,x)\big)^n(t-s)^n + \cdots . \end{aligned} \tag{6}$$

Note that $g(t,x)$ is the specified boundary condition for the partial differential equation. Indeed, $g(t,x) = f(x)$. Hence, we make this substitution and factor:

$$\begin{aligned} g(s,x) &= f(x) + \big((a+bx)\partial_x f(x) + \frac{c^2}{2}x\partial_x\partial_x f(x)\big)(t-s) +\\ &\quad \frac{1}{2}\big((a+bx)\partial_x f(x) + \frac{c^2}{2}x\partial_x\partial_x f(x)\big)^2(t-s)^2 + \cdots +\\ &\quad \frac{1}{n!}\big((a+bx)\partial_x f(x) + \frac{c^2}{2}x\partial_x\partial_x f(x)\big)^n(t-s)^n + \cdots\\ &= \big[1 + \big((a+bx)\partial_x + \frac{c^2}{2}x\partial_x\partial_x\big)(t-s) +\\ &\quad \frac{1}{2}\big((a+bx)\partial_x + \frac{c^2}{2}x\partial_x\partial_x\big)^2(t-s)^2 + \cdots +\\ &\quad \frac{1}{n!}\big((a+bx)\partial_x + \frac{c^2}{2}x\partial_x\partial_x\big)^n(t-s)^n + \cdots\big]\cdot f(x). \end{aligned} \tag{7}$$

Upon careful examination of Equation 7, it is evident that the operator term to the left of $f(x)$ is a power series representation of the exponential in $\big((a+bx)\partial_x f(x)+\frac{c^2}{2}x\partial_x\partial_x f(x)\big)(t-s)$. Hence, we define the exponential of the operator by the given power series and write

$$\begin{aligned} g(s,x) &= \exp\left[(t-s)\big((a+bx)\partial_x f(x)+\frac{c^2}{2}x\partial_x\partial_x f(x)\big)\right]\cdot f(x) \\ &= \exp\big[(t-s)H\big]\cdot f(x). \end{aligned} \tag{8}$$

□

Now that the solution to the CIR partial differential equation (Equation 1.3) has been characterized appropriately, this will help us to derive the density function of the random variable evolving according to the CIR stochastic differential equation. But first, the following definition will prove useful:

**Definition 2.3** (The CIR Distribution Operator)**.**

$$\widehat{x}(t) := e^{tH}xe^{-tH}$$

*where $H$ is the previously defined operator from Definition 2.1*

The next theorem will at last show how to find the distribution of $X_\cdot(t)$.

**Theorem 2.4** (The CIR Distribution Theorem)**.** *Suppose an interest rate process evolves according to the CIR Stochastic Differential Equation. Also suppose that the initial interest rate at time zero, $X_\cdot(0)$, is $x$. Then the density function of $X_\cdot(t)$ is given by the spectrum of the operator $\widehat{x}(t)$. More specifically, suppose*

$$\exists Q(\lambda,t)\ni \widehat{x}(t)\cdot Q(\lambda,t)=\lambda Q(\lambda,t)$$

$$\int_0^\infty Q(\lambda,t)d\lambda = 1.$$

*Then $Q(\lambda,t)$ is the density function of the random variable evolving according to the CIR stochastic differential equation at time $t$.*

As will be shown in the following proof, $Q(\lambda,t)$ turns out to be the Green's function of the CIR partial differential equation. Hence, the theorem above asserts, in essence, that the Green's function of the CIR partial differential equation gives the desired density function.

*Proof.* Consider the function $g(0,x)$. By Theorem 2.2,

$$g(0,x) = e^{tH}\cdot f(x). \tag{9}$$

Because $e^{tH}$ is a differential operator in $x$, $e^{tH}\cdot 1 = 1$. Likewise, $e^{-tH}\cdot 1 = 1$. Hence, $e^{-tH}\cdot e^{tH}\cdot 1 = 1$ and it will not hurt to add this factor of 1 at the end of Equation 9:

$$g(0,x) = \big[e^{tH}\cdot f(x)\big]\big[e^{-tH}\cdot e^{tH}\cdot 1\big]. \tag{10}$$

Now we make two very plausible assumptions: that the operators in Equation 10 are associative and that the function $f(x)$ is analytic. This yields:

$$\begin{aligned} g(0,x) &= \left[e^{tH} \cdot f(x)\right]\left[e^{-tH} \cdot e^{tH} \cdot 1\right] \\ &= \left[e^{tH} f(x) e^{-tH}\right] \cdot \left[e^{tH} \cdot 1\right] \\ &= \left[e^{tH} f(x) e^{-tH}\right] \cdot 1 \\ &= f\big(e^{tH} x e^{-tH}\big) \cdot 1 \\ &= f\big(\widehat{x}(t)\big) \cdot 1. \end{aligned} \tag{11}$$

Recall that under the assumptions of this theorem we are trying to prove, the function $Q(\lambda, t)$ is normalized to one. Hence:

$$\begin{aligned} g(0,x) &= f\big(\widehat{x}(t)\big) \cdot 1 \\ &= f\big(\widehat{x}(t)\big) \cdot \int_0^\infty Q(\lambda,t)d\lambda \\ &= \int_0^\infty f\big(\widehat{x}(t)\big) \cdot Q(\lambda,t)d\lambda \\ &= \int_0^\infty f\big(\lambda\big) Q(\lambda,t)d\lambda. \end{aligned} \tag{12}$$

Recall the definition of $g(s,x)$: (Equation 1.3)

$$g(s,x) := \mathbb{E}\big\{f\big(X.(t)\big)|X.(s) = x\big\}. \tag{13}$$

Therefore, we have just shown that:

$$\mathbb{E}\big\{f\big(X.(t)\big)|X.(0) = x\big\} = \int_0^\infty f\big(\lambda\big) Q(\lambda,t)d\lambda. \tag{14}$$

This directly implies that $Q(\lambda, t)$ is the density function of $X.(t)$ given that $X.(0) = x$. One simple case to check this is to plug in an indicator function for $f(\lambda)$. Furthermore, note that $Q(\lambda, t)$ is the Green's function of the CIR partial differential equation 1.3. Hence, we have shown that the Green's function to the CIR PDE is simply just the distribution of the random variable evolving according to the CIR stochastic differential equation. □

2.2. **Deriving the CIR Distribution Operator.** In order to compute the density function of $X.(t)$, we must derive the spectrum of $\widehat{x}(t)$. However, the operator as it is currently defined is difficult to work with. Therefore, we derive a new formula for $\widehat{x}(t)$ in terms of the constants appearing in the CIR stochastic differential equation (Equation 1.3). As usual, a new definition will prove convenient later on:

**Definition 2.5.**

$$\widehat{\partial_x}(t) := e^{tH} \partial_x e^{-tH}$$

**Theorem 2.6.** *The differential operator $\widehat{x}(t)$ is the unique solution to the system of operator-valued differential equations:*

$$\widehat{x}'(t) = a + b\widehat{x}(t) + c^2 \widehat{x}(t) \widehat{\partial_x}(t)$$

$$\widehat{\partial_x}'(t) = -b\widehat{\partial_x}(t) - \frac{c^2}{2}\widehat{\partial_x}^2(t)$$

*with the initial conditions:*

$$\widehat{x}(0) = x$$

$$\widehat{\partial_x}(0) = \partial_x.$$

*Proof.* First, note that $\partial_t e^{tH} = e^{tH}H$ and $\partial_t e^{-tH} = -He^{tH}$. This can be verified by directly differentiating the power series representation of the operators term-by-term. Hence,

$$\begin{aligned}
\widehat{x}'(t) &= e^{tH}Hxe^{-tH} + e^{tH}x(-H)e^{-tH} \\
&= e^{tH}(Hx - xH)e^{-tH} \\
&= e^{tH}[H,x]e^{-tH} \qquad (15)
\end{aligned}$$

where we have used the product rule of differentiation.

Because a commutator shows up in equation 15, we can use well-known commutator relations to compute this equation. More specifically,

$$\begin{aligned}
[H,x] &= [(a+bx)\partial_x + \frac{c^2}{2}x(\partial_x)^2, x] \\
&= [(a+bx)\partial_x, x] + \frac{c^2}{2}[x(\partial_x)^2, x] \\
&= \big((a+bx)[\partial_x, x] + [a+bx, x]\partial_x\big) + \frac{c^2}{2}\big(x[(\partial_x)^2, x] + [x,x](\partial_x)^2\big) \\
&= \big((a+bx)\cdot 1 + 0\cdot\partial_x\big) + \frac{c^2}{2}\big(x[(\partial_x)^2, x] + 0\cdot(\partial_x)^2\big) \\
&= (a+bx)\cdot 1 + \frac{c^2}{2}x\big(\partial_x[\partial_x, x] + [\partial_x, x]\partial_x\big) \\
&= (a+bx) + \frac{c^2}{2}x\big(\partial_x\cdot 1 + 1\cdot\partial_x\big) \\
&= a + bx + c^2x\partial_x. \qquad (16)
\end{aligned}$$

Now plugging this result back into equation 15,

$$\begin{aligned}
\widehat{x}'(t) &= e^{tH}[H,x]e^{-tH} \\
&= e^{tH}\big(a + bx + c^2x\partial_x\big)e^{-tH} \\
&= e^{tH}ae^{-tH} + e^{tH}bxe^{-tH} + e^{tH}c^2x\partial_x e^{-tH} \\
&= e^{tH}e^{-tH}a + be^{tH}xe^{-tH} + c^2e^{tH}x\partial_x e^{-tH} \\
&= a + be^{tH}xe^{-tH} + c^2e^{tH}xe^{-tH}e^{tH}\partial_x e^{-tH} \\
&= a + b\widehat{x}(t) + c^2\widehat{x}(t)\widehat{\partial_x}(t) \qquad (17)
\end{aligned}$$

As for the initial value, plugging zero directly into $\widehat{x}(t)$ yields simply $x$.

Now consider $\widehat{\partial_x}'(t)$. By the same reasoning as above,

$$\begin{aligned}
\widehat{\partial_x}'(t) &= e^{tH}H\partial_x + e^{tH}\partial_x(-H)e^{-tH} \\
&= e^{tH}H\partial_x - e^{tH}\partial_x(H)e^{-tH} \\
&= e^{tH}\big[(a+bx)\partial_x + \frac{c^2}{2}x(\partial_x)^2\big]\partial_x - e^{tH}\partial_x\big[(a+bx)\partial_x + \frac{c^2}{2}x(\partial_x)^2\big]e^{-tH} \\
&= e^{tH}\big[a\partial_x^2 + bx\partial_x^2 + \frac{c^2}{2}x\partial_x^3 - a\partial_x^2 - b(\partial_x + x\partial_x^2) - \frac{c^2}{2}(\partial_x^2 + x\partial_x^3)\big]e^{-tH} \\
&= e^{tH}\big[-b\partial_x - \frac{c^2}{2}\partial_x^2\big]e^{-tH} \\
&= e^{tH}\big[-b\partial_x\big]e^{-tH} - e^{tH}\big[\frac{c^2}{2}\partial_x^2\big]e^{-tH} \\
&= -be^{tH}\big[\partial_x\big]e^{-tH} - \frac{c^2}{2}e^{tH}\big[\partial_x^2\big]e^{-tH} \\
&= -b\widehat{\partial_x}(t) - \frac{c^2}{2}\widehat{\partial_x}(t). \qquad (18)
\end{aligned}$$

As for the initial value, once again it can be directly verified by plugging zero into $\widehat{\partial_x}(t)$. □

**Theorem 2.7** (Explicit Form for the CIR Distribution Operator)**.** *The solution to the system of differential equations in Theorem 2.6, and hence an explicit form for the differential operator $\widehat{x}(t)$ is:*

$$\widehat{x}(t) = \left[x(1 + \frac{c^2}{2}\frac{1-e^{-bt}}{b}\partial_x)^2 + a\frac{1-e^{-bt}}{b}(1 + \frac{c^2}{2}\frac{1-e^{-bt}}{b}\partial_x)\right]e^{bt}$$

*Note that this is a second-order ordinary differential operator.*

*Proof.* MISSING!! □

2.3. **Computing the Spectrum of the CIR Distribution Operator.** According to Theorem 2.4, the spectrum of the second-order differential operator $\widehat{x}(t)$ gives the distribution of the random variable evolving according to the CIR stochastic differential equation. Finding this spectrum involves solving a linear second-order ordinary differential equation as we now show.

**Theorem 2.8** (Second-Order Linear Ordinary Differential Equation)**.** *Define the following functions:*

$$\widehat{a}(x) := \frac{e^{bt}c^4(1-e^{-bt})^2}{4b^2}x$$

$$\widehat{b}(x) := \frac{ac^2e^{bt}(1-e^{-bt})^2}{2b^2} + \frac{e^{bt}c^2(1-e^{-bt})}{b}x$$

$$\widehat{c}(x) := \frac{ae^{bt}(1-e^{-bt})}{b} + e^{bt}x.$$

*The solution to the differential equation*

$$\widehat{a}(x)Q_{xx}(\lambda,t) + \widehat{b}(x)Q_x(\lambda,t) + (\widehat{c}(x) - \lambda)Q(\lambda,t) = 0$$

*is the eigenfunction of $\widehat{x}(t)$ for the eigenvalue $\lambda$.*

*Proof.* The eigenfunction of $\widehat{x}(t)$ for the eigenvalue $\lambda$ is given by:

$$\widehat{x}(t) \cdot Q(\lambda, t) = \lambda Q(\lambda, t). \tag{19}$$

Now we can simply plug-in the explicit form for $\widehat{x}(t)$ and simplify:

$$\Big[\big[x(1+\frac{c^2}{2}\frac{1-e^{-bt}}{b}\partial_x)^2 + a\frac{1-e^{-bt}}{b}(1+\frac{c^2}{2}\frac{1-e^{-bt}}{b}\partial_x)\big]e^{bt}\Big] \cdot Q(\lambda,t) = \lambda Q(\lambda,t)$$

$$\big[xe^{bt} + xe^{bt}c^2\frac{1-e^{-bt}}{b}\partial_x + xe^{bt}\frac{c^4(1-e^{-bt})^2}{4b^2}{\partial_x}^2 + \frac{ae^{bt}(1-e^{-bt})}{b} + \frac{ae^{bt}c^2(1-e^{-bt})^2}{2b^2}\partial_x\big] \cdot Q(\lambda,t) = \lambda Q(\lambda,t)$$

$$\big[xe^{bt} + xe^{bt}c^2\frac{1-e^{-bt}}{b}Q_x(\lambda,t) + xe^{bt}\frac{c^4(1-e^{-bt})^2}{4b^2}Q_{xx}(\lambda,t) + \frac{ae^{bt}(1-e^{-bt})}{b} + \frac{ae^{bt}c^2(1-e^{-bt})^2}{2b^2}Q_x(\lambda,t)\big] \cdot Q(\lambda,t) - \lambda Q(\lambda,t) = 0$$

$$\frac{e^{bt}c^4(1-e^{-bt})^2}{4b^2}xQ_{xx}(\lambda,t) + \big[\frac{ac^2e^{bt}(1-e^{-bt})^2}{2b^2} + \frac{e^{bt}c^2(1-e^{-bt})}{b}x\big]Q_x(\lambda,t) + (\frac{ae^{bt}(1-e^{-bt})}{b} + e^{bt}x. - \lambda)Q(\lambda,t) = 0 \tag{20}$$

$$\widehat{a}(x)Q_{xx}(\lambda,t) + \widehat{b}(x)Q_x(\lambda,t) + (\widehat{c}(x) - \lambda)Q(\lambda,t) = 0.$$

Note that all steps given above are reversible. Hence, $Q(\lambda, t)$ is the eigenfunction of $\widehat{x}(t)$ for the eigenvalue $\lambda$. □

Now, it remains to solve the linear second-order ordinary differential equation given in Theorem 2.8 to find the distribution of the random variable evolving according to the CIR stochastic differential equation. The solution to the linear second-order ordinary differential equation can be obtained by standard series techniques. Rather than provide the tedious details to this process, we simply state the solutions, which can be easily checked by simply plugging it back into the differential equation:

**Theorem 2.9** (The Solution to the Differential Equation in Theorem 2.8)**.** *Since the differential equation is second-order, there are two linearly-indenpendent solutions to the equation in Theorem 2.8:*

$$Q_1(\lambda,t) = e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}I_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})$$

$$Q_2(\lambda,t) = e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}K_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})$$

*where* $\eta = \frac{2a}{c^2} - 1$, $\zeta = \frac{2b}{c^2(e^{bt}-1)}$, *and* $I_\eta(z)$ *and* $K_\eta(z)$ *are modified Bessel Functions of order* $\eta$ *of first and second kind, respectively.*

Note that because we are working with a linear differential equation, any linear combination of $Q_1(\lambda, t)$ and $Q_2(\lambda, t)$ is still a solution to the differential equation. That is, $C_1Q_1(\lambda, t) + C_2Q_2(\lambda, t)$ is a solution to the differential equation given in Theorem 2.8. Also note that the constants $C_1$ and $C_2$ may both be a function of the variables $\lambda$, $t$, $a$, $b$, and $c$, but may *not* be a function of x. We summarize all of this with a corollary:

**Corollary 2.10** (The General Solution)**.** *The most general solution to the second-order linear differential equation given in Theorem 2.8 is:*

$$C_1(\lambda,t,a,b,c)e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}I_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})+C_2(\lambda,t,a,b,c)e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}K_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})$$

2.4. **Computing the Distribution of the Interest Rate.** Although we have found the eigenfunctions of $\widehat{x}(t)$, they still must be normalized as according to Theorem 2.4 to obtain the distribution of the interest rate random variable evolving according to the CIR stochastic differential equation. We may also perform some analysis on the two independent solutions $Q_1(\lambda,t)$ and $Q_2(\lambda,t)$ to see if they indeed behave in an expected way. Indeed, both $Q_1(\lambda,t)$ and $Q_2(\lambda,t)$ look identical except that $Q_1(\lambda,t)$ contains a modified Bessel function of the first kind, while $Q_2(\lambda,t)$ contains a modified Bessel function of the second kind. Hence, any behavioral differences between these two functions must arise out of a difference between the two modified Bessel functions. Perhaps the most striking difference is that $I_\eta(z)$ takes a finite value (mostly zero) at $z=0$, while $K_\eta(z)$ diverges at $z=0$. This simply allows us to eliminate $Q_2(\lambda,t)$ from being a distribution function:

**Theorem 2.11** (Distribution Function Form)**.** *The distribution function of the random variable evolving via the CIR stochastic differential equation must take the form of*

$$Q(\lambda,t,x)=C_1(\lambda,t,a,b,c)e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}I_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})$$

*where* $\eta=\frac{2a}{c^2}-1$, $\zeta=\frac{2b}{c^2(e^{bt}-1)}$, *and* $I_\eta(z)$ *is the modified Bessel Function of order* $\eta$ *of first kind.*

*Proof.* We know that the *most* general way of expressing the distribution function is the general solution to the differential equation in Theorem 2.8 as given in Corollary 2.10:

(21)
$$C_1(\lambda,t,a,b,c)e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}I_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})+C_2(\lambda,t,a,b,c)e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}K_\eta(2\sqrt{e^{bt}\lambda x\zeta^2}).$$

Now we seek to show that $C_2(\lambda,t,a,b,c)=0$.

We know that if we take the limit as the distribution function approaches zero in $x$, we expect that distribution function to be finite. Indeed, there is nothing special about setting $x$, the initial interest rate at time zero, to zero. Hence:

(22)
$$\lim_{x\to 0}\left[C_1(\lambda,t,a,b,c)e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}I_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})+C_2(\lambda,t,a,b,c)e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}K_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})\right]$$
$$=C_1(\lambda,t,a,b,c)\lim_{x\to 0}\left[e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}I_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})\right]+C_2(\lambda,t,a,b,c)\lim_{x\to 0}\left[e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}K_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})\right]$$

where we moved the limits in because the constants have no dependence on $x$.

Let's consider the limit in the second term of equation 22:

$$\lim_{x\to 0}\left[e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}K_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})\right] \tag{23}$$

Notice that the first two factors, $e^{-\zeta e^{bt}x}$ and $x^{-\frac{\eta}{2}}$ approach one and zero, respectively, as $x$ approaches zero. However, the third term $K_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})$ grows without bound as $x$ approaches zero because the argument inside the Bessel function approaches zero. This means that convergence of the entire limit is not guaranteed because we have a $0\cdot\infty$ situation. And multiplication by the undetermined coefficient $C_2(\lambda,t,a,b,c)$ cannot fix this because the coefficient has no $x$ dependence.

So, unless we restrict the constants $a$, $b$, $c$ and the variables $t$ and $\lambda$ to very specific values, we can see that the function $Q_2(\lambda,t) = e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}K_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})$ is not a function that behaves like a distribution function. Indeed, the function blows up when the initial interest rate at time zero is set to zero for certain values of $a$, $b$, $c$, $t$ and $\lambda$.

On the other hand, $Q_1(\lambda,t) = e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}I_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})$ always has a finite limit at $x = 0$ because the Bessel function $I_\eta(2\sqrt{e^{bt}\lambda x\zeta^2})$ has a finite value at $x = 0$. □

Now it remains to find the explicit form of $C_1(\lambda,t,a,b,c)$. Note that we still cannot impose the normalization condition: $\int_0^\infty Q(\lambda,t,x)d\lambda$ because we do not known the explicit dependence on $\lambda$ yet. On the other hand, we have fixed the $x$ dependence of the density function because the undetermined factor $C_1(\lambda,t,a,b,c)$ has no $x$ dependence.

Note that our density function, or what we have of it so far with the undetermined factor, is the Green's function of the CIR partial differential equation (equation 1.3). This is because it was one of the independent solutions of the ordinary differential equation in Theorem 2.8. Therefore, it is a solution to the CIR partial differential equation. In other words:

**Theorem 2.12.** *Consider our distribution function*

$$\begin{aligned} Q(\lambda,t,x) &= C_1(\lambda,t,a,b,c)e^{-\zeta e^{bt}x}x^{-\frac{\eta}{2}}I_\eta(2\sqrt{e^{bt}\lambda x\zeta^2}) \\ &=: C_1(\lambda,t)q(\lambda,t,x) \end{aligned}$$

*$Q(\lambda,t,x)$ is a solution to the CIR partial differential equation (equation 1.3). In other words,*

$$Q_t(\lambda,t,x) = H\cdot Q(\lambda,t,x).$$

*Proof.* Because $Q(\lambda,t,x)$ is one of the independent solutions of the second-order ordinary differential equation appearing in theorem 2.8, it is a Green's function to the CIR partial differential equation. Hence, it is a solution to the CIR partial differential equation. □

Because $H$ is a differential operator in $x$, it does not act on the term $C_1$ of $Q$. Hence, plugging in $Q(\lambda,t,x)$ directly into CIR partial differential equation yields the following ordinary differential equation for $C_1$:

**Equation 2.13** (Computing $C_1(\lambda,t)$)**.**

$$C_{1t}(\lambda,t) = C_1(\lambda,t)\Big(\frac{(a+bx)q_x + \frac{c^2}{2}xq_{xx} - q_t}{q}\Big)$$

*where the inputs of $q(\lambda,t,x)$ have been suppressed.*

Note that even though factors and derivatives of $x$ appear in the above equation, we expect all of them to cancel out at some point because $C_1(\lambda,t)$ has no $x$ dependence. Furthermore, because the differential equation is in $t$, this will finally fix the $t$ dependence of the distribution.

**Theorem 2.14** (The Solution to Equation 2.13)**.**

$$C_1(\lambda,t) =$$